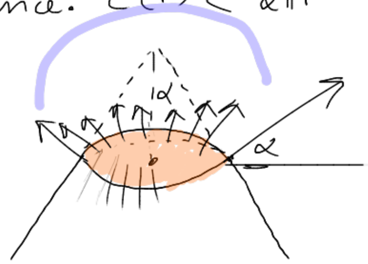
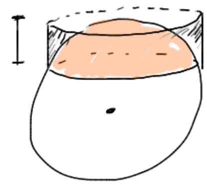
Curvature of a spike —— Polyhedral Theorema Egregium

circumference: $c(r) < 2\pi r$

cone



$1-\sin(\alpha)$



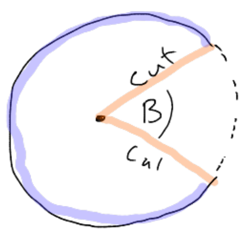
S^1

blunt tip, then compute curvature as usual using Gauss's Spherical map

K(spike) = total K of blunted tip = area of polar cap (spherical image)
extrinsic

$K(\text{spike}) = 2\pi(1-\sin\alpha)$
(intrinsic)

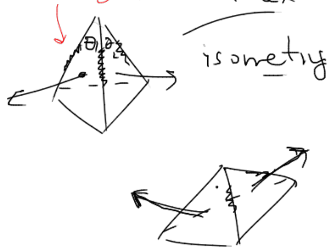
$\sin\alpha$



$2\pi\sin(\alpha) = 2\pi - B$

$B = 2\pi(1-\sin\alpha) = K(\text{spike})$

hinged

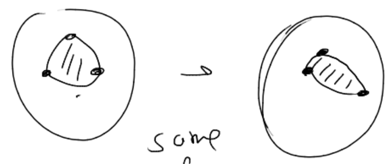


Flex

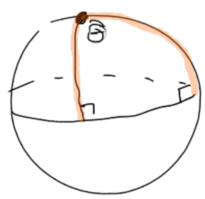
isometry

spherical projection of normals

has constant area through flexing (isometry)

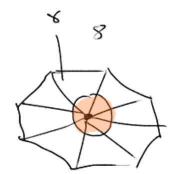


same angles



... area of spherical polygon is determined solely by it's angles.

$$K(\text{spike}) = A(P_m) = \text{angle sum} + \underbrace{(m-2)\pi}_{\text{Eucl.}}$$



Total Angle sum,

$8(2\gamma)$

$8(\pi - \frac{\pi}{4})$

$8\pi - 2\pi$ 6π

$2\gamma + \frac{2\pi}{8} = \pi$

$\frac{2\pi}{8}$

$2\gamma = \pi - \frac{\pi}{4}$

$\gamma = \frac{\pi}{2} - \frac{\pi}{8}$

Shape Operator

we measure the shape of curves w/ curvature & torsion, Functions —

the analagous measurement device for surfaces (2-D), M, is linear operator defined on each tangent space of the surface.

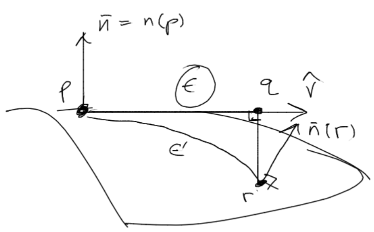
$S: T_pM \to T_pM$.

the algebraic invariants have geometric meaning —

In contrast to computing extrinsic curvature (how much normals spread)

— we compute (via S)
how fast a normal vector changes as you move in a specific direction, $\hat{v}$

unit length $|\hat{v}| = 1$

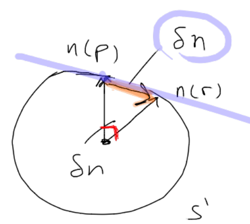


For small ε
move from p in direction of $\hat{v}$,
by distance ε

$pr \approx \epsilon$

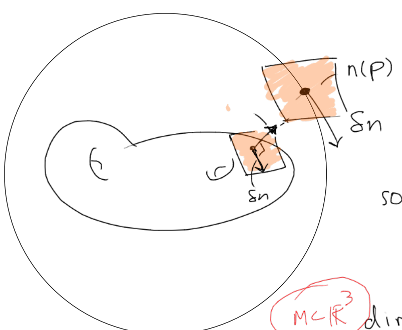
$\delta n \equiv \bar{n}(r) - \bar{n}(p)$

think: both as images of spherical map



this matters b/c the tangent plane, @ the tip of n(p) on S' is parallel to T_pM.

For small δn
the difference vector is ultimately ⊥ to n(p), thus δn is parallel to a tangent vector @ the tip of n(p).



so ——

$$\nabla_{\hat{v}} n = \lim_{\epsilon \to 0} \frac{\delta n}{\epsilon}$$

$M \subset \mathbb{R}^3$ directional derivative

M, no embedding
covariant derivative

so ..., $\nabla_{\hat{v}} n \approx \frac{\delta n}{\epsilon}$

$\Rightarrow$

$\epsilon \nabla_{\hat{v}} n \approx \delta n$

$f'(cx) = cf'(x)$

$\nabla_{\epsilon\hat{v}} n \approx \delta n$

how fast & in what direction does n move, as we move on the surface in the direction of v by a tiny amount

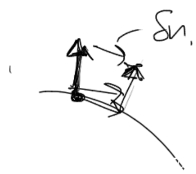
Note: If $\bar{v} = c\hat{v}$ velocity then $\nabla_{\bar{v}} n = \nabla_{c\hat{v}} n = c\nabla_{\hat{v}} n$ represents a change in $\bar{n}$ wrt time.

think:

If $\bar{v}$ is small then $\bar{v} = \epsilon \hat{v}$

$$\nabla_{\bar{v}} n = \nabla_{\epsilon \hat{v}} n = \epsilon \nabla_{\hat{v}} n \quad \text{so} \quad \underline{\nabla_{\hat{v}} n} \simeq \delta n$$

change in $\bar{n}$ from tip of v to tail of v.



Finally Shape Operator

$$S : T_pM \to T_pM$$

$$S(\bar{v}) = -\nabla_{\bar{v}} \bar{n}$$

the artificial minus sign introduced here, drastically reduces the number of minus signs encountered later

The directional deriv. $\nabla_v n$ tells us how tangent planes of M vary in the v direction —

this gives us an infinitessimal notion of how M curves in $\mathbb{R}^3$.

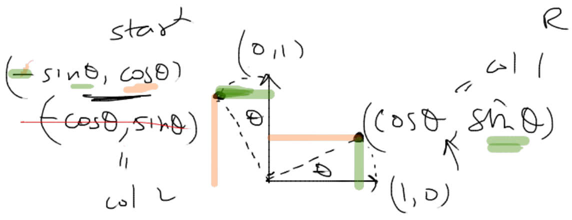
Remark: $S(a\bar{v} + b\bar{w}) = -\left[\nabla_{a\bar{v}+b\bar{w}} n\right] = -\left[\nabla_{a\bar{v}} n + \nabla_{b\bar{w}} n\right]$

$$= -a\nabla_{\bar{v}} n - b\nabla_{\bar{w}} n$$

$$= a(-\nabla_v n) + b(-\nabla_w n)$$

$$= aS(\bar{v}) + bS(\bar{w})$$

Linear Operators have matrix representations we'll need - expression of a rotation in $\mathbb{R}^2$ by θ as a matrix



basis

$$\text{Rotation} = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

matrix rep. Lin. op
$\Rightarrow$ col i = image of basis elt. i

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

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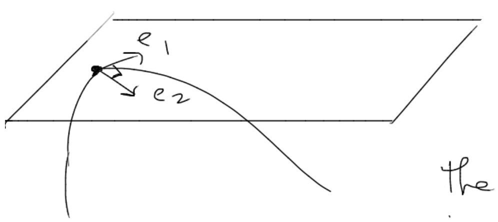
geometry of S



Let $\{e_1, e_2\}$ be two unit vectors pointing in principal direction. (orthogonal)

the principal directions are the eigenvectors of the Shape Operator.

principal values are eigenvalues

So in this basis, $(1,0) \leftrightarrow$ principal direction

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} k_1 \\ 0 \end{pmatrix}$

$$[S] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$