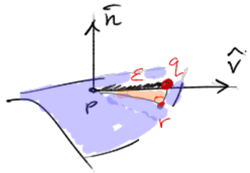
The Shape Operator

We measure shape of curves w/ curvature and torsion functions
- the analagous measurement device for surfaces $M \subset \mathbb{R}^3$ is a linear operator defined on each of the tangent planes of M.
- The algebraic invariants of a surface's Shape Operator have geometric meanings.

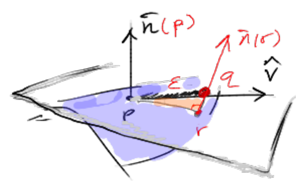
In contrast to computation of extrinsic curvature, how much normals spread in small patch
- we compute how fast a normal changes as we move in some direction, $\hat{v}$ ($|\hat{v}| = 1$)

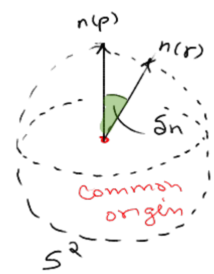


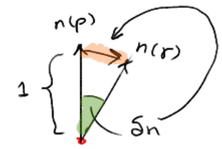
For small ϵ

$pr \approx \epsilon$

thus $\delta n \equiv n(r) - n(p)$







δn is difference vector and is ultimately perpendicular to $n(p) = \hat{n}$ (parallel to T_p)

$\delta n \in T_p$

directional derivative

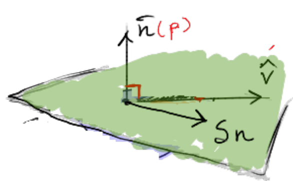
$$\nabla_{\hat{v}} n \equiv \lim_{\epsilon \to 0} \frac{\delta n}{\epsilon}$$

per unit length ($|\hat{v}| = 1$)

more generally ...

if $\bar{v} = c \cdot \hat{v}$, then $\nabla_{\bar{v}} \bar{n} = \nabla_{c \cdot \hat{v}} \bar{n} = c \nabla_{\hat{v}} \bar{n}$
(say velocity)

represents change in $\bar{n}$ wrt time since $\hat{v}$ is unit length



thus $\nabla_{\hat{v}} n \approx \frac{\delta n}{\epsilon} \Leftrightarrow \epsilon \nabla_{\hat{v}} n = \delta n \Leftrightarrow \nabla_{\epsilon \hat{v}} \bar{n} \approx \delta n$

If v is small, i.e., $v = \epsilon \hat{v}$ then $\nabla_v \bar{n} \approx \delta n$ [change in $\bar{n}$ from tail to tip of v.]

shape operator

$$S(\bar{v}) = -\nabla_{\bar{v}} \bar{n}$$

this artificial minus sign drastically reduces # of minus signs later the directional derivative $\nabla_v n$ tells how the tangent planes of M are varying in the v direction —

— and this gives an *infinitessimal* description of the way M is curving in $\mathbb{R}^3$.

Prop the shape operator is a linear operator

$$S(a\vec{v} + b\vec{w}) = -\nabla_{(a\vec{v} + b\vec{w})} \hat{n} = -\nabla_{a\vec{v}} \hat{n} - \nabla_{b\vec{w}} \hat{n}$$

$$= -a\nabla_{\vec{v}} \hat{n} - b\nabla_{\vec{w}} \hat{n} = aS(v) + bS(\vec{w})$$

... given by a matrix ...

jth column = image of j-th basis vector

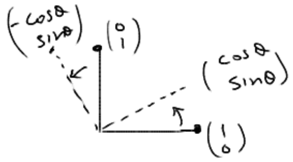
O'neill's def'n of directional deriv.

Def'n Let $v \in T_pM$, $f \in C^1(M)$

the derivative $v[f]$ of f wrt v is the common value of $\frac{d}{dt}\left(f(\alpha(0))\right)$ for all curves α in M w/ initial velocity v.

$\nabla_v Z = Z_p'(0)$ (covariant derivative)

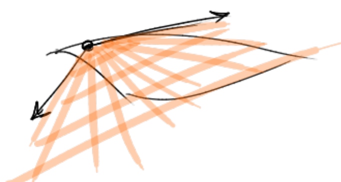
Represent Rotation, by θ in $\mathbb{R}^2$ via matrix : $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



the geometric effect of S

Let e_1, e_2 be two unit (orthog) vectors along first & second principal directions.

use as basis for T_p.



The principal directions are the eigenvectors of the Shape Operator, & principal curvatures are corresp. eigenvalues

$$S(e_i) = k_i e_i$$

$\Rightarrow$ S stretches in principal directions, and stretches a general vector in both $\perp$ directions

i.e., Matrix $[S] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

the general Matrix

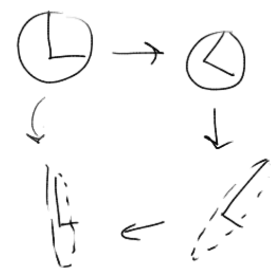
shape operator : geometric object ... its matrix depends on basis

in basis E_1, E_2, not e_1, e_2 just conjugate matrix by rotation that takes $E_i \mapsto e_i$

$\{E_1, E_2\} \to [S_1]$ $= A$ matrix (stretch)

$R_\theta A R_\theta^{-1} \leftrightarrow \{e_1, e_2\}$

rotate by θ — stretch — rotate by $-\theta$



$$[S] = R_\theta \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} R_{-\theta}$$

$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} k_1 c^2 + k_2 s^2 & (k_1 - k_2) sc \\ (k_1 - k_2) sc & k_1 s^2 + k_2 c^2 \end{pmatrix}$$

Euler's Formula $= \kappa(E_1) =$ 1st principal curvature

thm If $\hat{v}$ is arb. unit tangent vector, then the curvature $\kappa(\hat{v})$ of the normal section in this direction is

$$\kappa(\hat{v}) = \hat{v} \cdot S(\hat{v}) = \text{projection of } S(\hat{v}) \text{ onto direction of } \hat{v}.$$

i.e., the normal curvature

(so we can also compute the geodesic curvature from this and the curvature vector itself.)

ex

$$\nabla_{\hat{v}} \hat{v} = -Sn = -\frac{dn}{ds} = -(-\kappa(\hat{v}) \cdot n) = \kappa(\hat{v}) n \quad \Rightarrow$$

($\hat{v}$ = unit velocity along normal section)

$$0 = \nabla_{\hat{v}}(\hat{v} \cdot \bar{n}) = \underbrace{\nabla_{\hat{v}}(\hat{v}) \cdot \bar{n}}_{\kappa(\bar{v})\underbrace{\bar{n}\cdot\bar{n}}_{=1}} + \hat{v} \cdot \nabla_{\hat{v}}(\bar{n}) = \kappa(\bar{v}) - \hat{v} \cdot S(\bar{v})$$

so $\kappa(\bar{v}) = \hat{v} \cdot S(\bar{v})$

Other simple expressions

$$[S] = \begin{bmatrix} E_1 \cdot S(E_1) & E_1 \cdot S(E_2) \\ E_2 \cdot S(E_2) & E_2 \cdot S(E_2) \end{bmatrix} = \begin{bmatrix} K(E_1) & \frac{\Delta K}{2} \sin 2\theta \\ \frac{\Delta K}{2} \sin 2\theta & K(E_2) \end{bmatrix}$$

ex) S^2_R

As $\bar{n}$ moves away from any point p in direction of $\bar{v}$.

$\bar{n}$ topples forward in direction of $\bar{v}$.

$S(v) = -cv$ for some c. Every direction is principal so

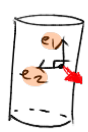
$S(v) = -\frac{1}{R} v$. Shape Op. $\equiv$ mult. by $\frac{-1}{R}$ (const.)

ex) Π = plane in $\mathbb{R}^3$, Let $\{n\}$ be normal to Π.

Note $\delta n = 0$ as we move normals around on Π.

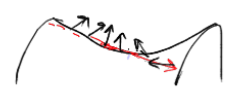
thus $S(n) = 0$

ex) $x^2 + y^2 = r^2$ in $\mathbb{R}^3$



$S(e_1) = 0$

$S(e_2) = \frac{-1}{r} \cdot e_2$



ex) Saddle Surface $z = xy$

① @ $p = (0,0,0)$

② note: x-axis, y-axis $\subset$ Surface

③ $\Rightarrow u_1 = (1,0,0)$, $u_2 = (0,1,0)$ $\in T_p$

④ $n = (0,0,1)$
as n moves along x-axis – swings L–R

$\nabla_{u_1} n = -u_2$

$\nabla_{u_2} n = -u_1$

⑤ $S(au_1 + bu_2) = bu_1 + au_2$

Curves

1-dimension

κ, τ measure rate of change of T & B (hence N)

Surface

only 1 unit vector is determined – normal

@ each point $\exists$ whole plane of directions n can move so rates of change aren't #'s but vectors