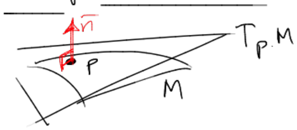
Wk 7 - Mon

Next Week, Due Mon:
ch. 20, 7, 15, 19

Today: ① Shape Operator
② Topology
③ GGB

Shape Operator:



$S : T_pM \to T_pM$, $S_p(\bar{v}) = \nabla_{v}\bar{n}$

S, being a map on tangent space, stretches in the principal directions:

$$[S] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

matrix representing S

k_i = principal curvatures

(choose the principal directions as basis for T_pM)

the general matrix (when you dont have prin vectors as basis —

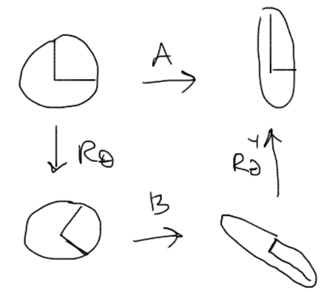
$\{e_1, e_2\} \equiv$ principal vectors

$\{E_1, E_2\} =$ chosen basis, determine $[S] = A$ ← matrix

Q: how to represent $[S]$ in the principal basis?

A: $B = R_\theta A \cdot R_\theta^{-1}$

conjugation by a rotation.



$$[S] = R_\theta \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix} R_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$c = \cos\theta$

$s = \sin\theta$

$$= \begin{pmatrix} k_1 c^2 + k_2 s^2 & (k_1 - k_2) sc \\ (k_1 - k_2) sc & k_1 s^2 + k_2 c^2 \end{pmatrix}$$

Need to know k_1, k_2 principal curvatures

θ, the angle b/w your basis & principal one.

thm. If $\hat{v}$ is arb. unit tangent vector, then

$\left(\hat{v} = \dfrac{\bar{v}'}{|\bar{v}'|}\right)$ then $|\hat{v}| = \left|\dfrac{\bar{v}'}{|\bar{v}'|}\right| = \dfrac{|\bar{v}'|}{|\bar{v}'|} = 1$

Then the curvature $k(\bar{v})$ of the normal section in this direction.

$$\boxed{k(\bar{v}) = \hat{v} \cdot \underbrace{S(\hat{v})}_{\text{vector} \atop {\shortparallel \atop [S]\hat{v}}}}$$

projection of $S(\hat{v})$ onto direction of $\hat{v}$.

(also, this allows us to compute the geodesic curvature & curvature vector itself)

ex

$$0 = \nabla_{\hat{v}}(\hat{v} \cdot \bar{n}) = \underbrace{\nabla_{\hat{v}} \hat{v}}_{(k(\hat{v}) \cdot n)} \cdot n + \hat{v} \cdot \underbrace{\nabla_{\hat{v}} \bar{n}}_{-S(\hat{v})}$$

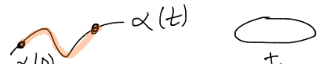
$\dfrac{dT}{ds} = k\bar{n}$

$$= k(\hat{v})\,\underbrace{\bar{n} \cdot \bar{n}}_{1} + \hat{v}$$

$$= k(\hat{v}) + \hat{v}(-S(\bar{v}))$$

$\Rightarrow$ $\boxed{k(\hat{v}) = \hat{v} \cdot S\hat{v}}$

How to get a unit speed parametrization



$\alpha(t)$

$\alpha(0)$

(parametrize by arc length)

arc length $= s(t) = \int_0^t |\alpha'(u)|\,du$ from $p = \alpha(0)$ up to $\alpha(t)$

solve for t... (you can, Inverse Function thm)

$s = A(t) - A(0)$ so $s - A(0) = A(t)$

FTC

$A^{-1}(s - A(0)) = t$

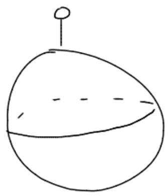
Note: $\dfrac{ds}{dt} = |\alpha'(t)| \Rightarrow dt = \dfrac{1}{|\alpha'(t)|}\,ds$

$$\left|\frac{d\alpha}{ds}\right| = \left|\frac{d\alpha}{dt} \cdot \frac{dt}{ds}\right| = |\alpha'(t)| \cdot \frac{\frac{1}{|\alpha'(t)|}\,ds}{ds} = 1$$

Compute some shape operators

$S(\bar{v}) = -\nabla \bar{n}$

ex S^2_R as $\bar{n}$ moves forward from any point in direction of $\bar{v}$.
$\bar{n}$ topples forward in direction of $\bar{v}$, (same!)



$S(v) = cv$, every direction is principal !!!

$[S] = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix}$

$S(v) = -\frac{1}{R} v$

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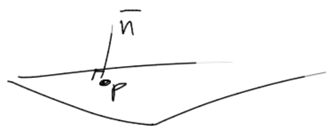
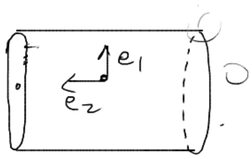
ex Π = plane in $\mathbb{R}^3$

$[S] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\nabla_v \bar{n} = 0$



ex



$s(e_1) = f(5,2) \cdot e_1$

$s(e_2) = 0$

in fact

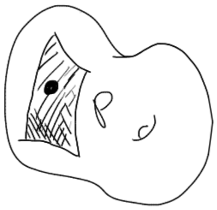
$[S] = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$

computation $= 0$

$5x^2 + 2y^2 = 10$

Ch. 16

Global Gauss Bonnet Theorem:

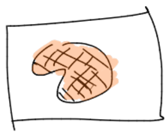


Total Curvature:

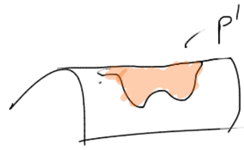
$$K(P) = \iint_P \kappa \, dA$$

Gaussian Curvature @ each point

Theorem Egregium:



$\xrightarrow{\text{isometry}}$

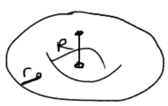


$K(P) = 0$ $K(P) = 0$

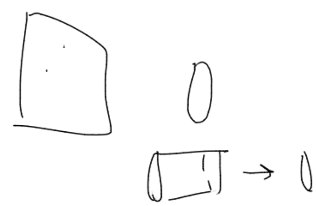
S^2_R

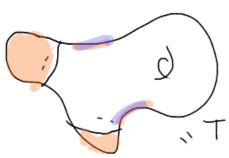
total curvature $= \iint_{S^2} 1/R \, dA = \frac{1}{R} \iint_{S^2} dA = \frac{1}{R}(4\pi r^2) = 4\pi$

$T^2_{R,r}$



total curvature $\iint_{T^2} \underbrace{\frac{1}{r(r+R\sec(\alpha))}}_{K} dA = 0$





$K(T) = 0$ (G.G.B.)

$\Rightarrow K = 0$