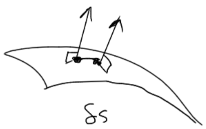
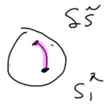
## Gauss Bonnett theorem

$K(\Sigma_g) = \iint_{\Sigma_g} K\, dA = 2\pi(2-2g) = 2\pi\chi(\Sigma_g) = 4\pi(1-g) = 4\pi \cdot \deg(n(\Sigma_g))$

$\downarrow$ degree; $n(\Sigma_g)$: spherical map of $\Sigma_g$




$K$ = rate of spread of tan. vector (normal)

$\delta\tilde{s} \approx K\,\delta s$

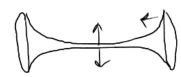
let $p$ roam over surface, track $K$

$\iint_{\Sigma_g} K\, dA$

For spheres & tori, genus $g$ surfaces we've seen this $= 2\pi\chi(\Sigma_g)$

(ex) $\Sigma_g$ =  = connected sum 

curvature is additive



$K(\Sigma_g) = g \cdot K(\text{torus}) + (g-1)\, K(\text{tube})$

$= g \cdot 0 + (g-1)(-4\pi) = 4\pi(1-g)$ degree of sp. map.

$= 2\pi\underbrace{(2-2g)}_{\chi(\Sigma_g)}$

## degree of Spherical Map

$n: \Sigma_g \to S_1^2$ ← radius

assumptions: $\Sigma_g$ closed, oriented, $\tilde{p} \in S_1^2$.

$\deg(n(\Sigma_g), \tilde{p}) \equiv P(\tilde{p}) - N(\tilde{p})$ $\quad$ $(\deg: \Sigma_g \to \mathbb{Z})$

$N(\tilde{p})$: the number of preimages of p-twiddle coming from negatively curved areas

$\deg(n(\Sigma_g), \tilde{p})$: degree of the normal map with respect to $\tilde{p}$

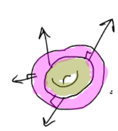
$P(\tilde{p})$: the number of preimages of p-twiddle coming from positively curved areas

the algebraic count of the number of times the normal map covers p-twiddle, counted with orientation.

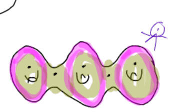
(ex)  $S_R^2$ $\quad \deg(n(S_R^2), \tilde{p}) = 1 - 0$ $\quad (1-g)$

└ cover once, & the $S_R^2$ is +-curved

(ex)  $\Sigma_1$ $\quad \deg(n(\Sigma_1), \tilde{p}) = 1 - 1 = 0$ $\quad (1-g)$




(ex)



$\deg(n(\Sigma_3), \tilde{p}) = P - N$ $\quad (1-g)$

$= 3 - 5 = -2$

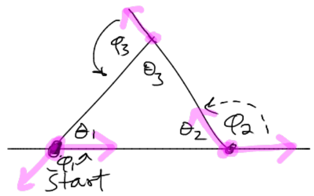

(presumably) has to do with curvature

$\boxed{\deg(n(\Sigma_g), \tilde{p}) = 1-g}$

$1-g$: topological invariant

# Heuristic Proof of GGBT

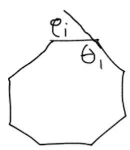
1st Plane Curves



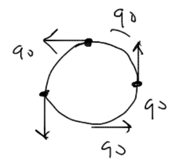

keep track of net change of direction $= \varphi_2 + \varphi_3 + \varphi_1 = \pi - \theta_2 + \pi - \theta_3 + \pi - \theta_1 = 2\pi$

this generalizes "greatly"

polygons: net rotation..




$\sum \varphi_i = 2\pi$




$\approx 2\pi$

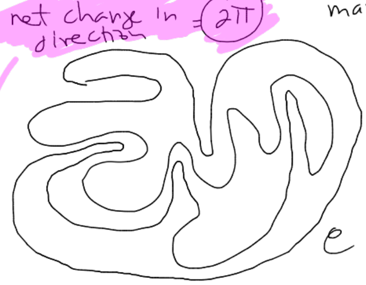
simple closed curve

no self intersections

loop

1-D manifold

net change in direction $= 2\pi$



$e$

Hopf's Umlaufsatz

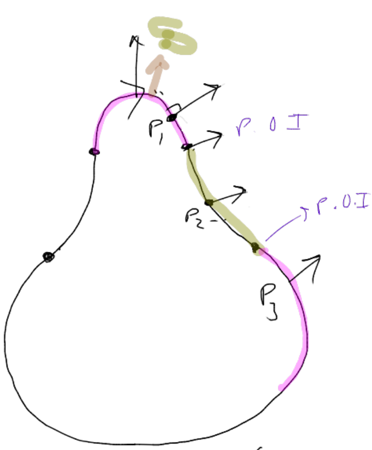
circulation theorem.

$$\oint_e K\,ds = \oint \frac{d\varphi}{ds}\,ds = \oint_e d\varphi = 2\pi \text{ net rotation of tangent}$$

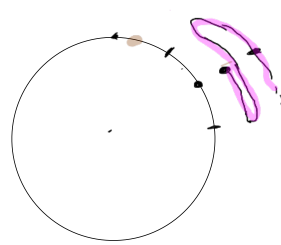
$$= \oint_e \frac{d\tilde{s}}{ds}\,ds = \int_{n(e)} d\tilde{s} = 2\pi \left(\begin{smallmatrix}\#\text{ of} \\ \text{times} \\ \text{wrap} \\ \text{around}\end{smallmatrix}\right)$$

$$\int_{S^1} ds = 2\pi$$

ex




$\xrightarrow{n}$



keep track of path like this (lift path off $S^1$)

$\tilde{p}$ covered 3 times

deg $= 2 - 1 = 1$

The same idea holds for surfaces.