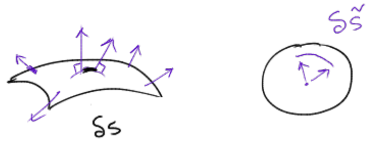
Gauss - Bonnett theorem

$$K(\Sigma_g) = \iint_{\Sigma_g} k\, dA = 2\pi \chi(G) = 2\pi(2-2g) = 2\pi \cdot 2(1-g) = 4\pi(1-g) = 4\pi \deg(n(\Sigma_g))$$



$\delta\tilde{s} \simeq k\,\delta s$ since k = rate of spread of tang. (normal) vectors

$= \dfrac{\delta\tilde{s}}{\delta s}$ (defined in nbhd of a point p)

let p roam over surface Σ_g, track $k = k(p)$ as we go:

$$\iint_{\Sigma_g} k\, dA$$

For sphere's and tori, genus g we've seen this equals $= 2\pi\chi(\Sigma_g)$

ex Σ_g =  = (connected sum) 

[curvature is additive]

$K(\Sigma_g)$ ═══ $g \cdot K($$) + (g-1)\cdot K($$)$

═══ $g(0) + (g-1)(-4\pi) = 4\pi - 4\pi \cdot g = 4\pi(1-g)$

Topological Degree of Spherical Map: $n : \Sigma_g \to S^2_1$

(assumption: Σ_g closed, oriented, $\tilde{p} \in S_1^2$)

$$\deg(n(\Sigma_g), \tilde{p}) \equiv P(\tilde{p}) - N(\tilde{p})$$

$P(\tilde{p})$: # preimages w/ $K>0$

$N(\tilde{p})$: # preimages w/ $K>0$

algebraic count of number of times $n(\Sigma_g)$ covers $\tilde{p}$, counting orientation

ex  S^2_R $\deg(n(\Sigma_g), \tilde{p}) = 1$

ex  Σ_1 $\deg(n(\Sigma_g), \tilde{p}) = 1 - 1 = 0$

ex  Σ_g $\deg(n(\Sigma_g), \tilde{p}) = \underbrace{g}_{\text{\# positive curvature covers}} - \underbrace{(g + g-1)}_{\text{\# negative}} = 1-g$

$\deg(n(\Sigma_g), \tilde{p}) = 1 - g$

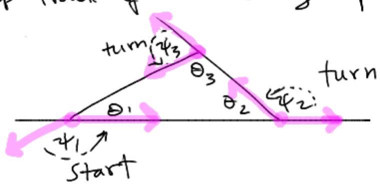
presumably having to do with curvature, i.e., rate of spread of normal

NEXT: show $1 - g = \frac{1}{2}\underbrace{\chi(\Sigma_g)}_{\text{topological invariant}}$

Ch.17 Heuristic proof of GGBT

Plane loop:

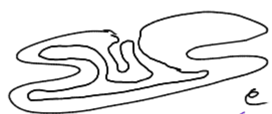
Keep track of net change of direction



net rotation: $\psi_2 + \psi_3 + \psi_1 = \pi - \theta_2 + \pi - \theta_3 + \pi - \theta_1 = 2\pi$ (since $\theta_1 + \theta_2 + \theta_3 = \pi$)

For n-gons:

net rotation: $\sum_{i=1}^{n} \psi_i = 2\pi$ since $\sum \theta_i = (n-2)\pi$

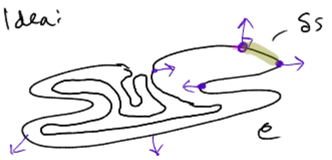
But what about 

? ['simple' loop]

Turns out: — Hopf's Umlaufsatz —

$$\oint_c \kappa\, ds = \int_c d\psi = \text{net rotation of } \gamma = 2\pi$$

Idea:



Normals cover S^1 = circle algebraically — once.

Over δs the directions of normal vector are spread over angle $\delta\psi$ $\Rightarrow$ arc of length $\delta\tilde{s} = \delta\psi$ on S^1.

i.e., $\kappa \equiv \frac{\delta\psi}{\delta s} \asymp \frac{\delta\tilde{s}}{\delta s}$

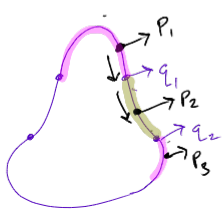
$$\Rightarrow \oint_c \kappa\, ds = \oint_c \frac{d\tilde{s}}{ds}\, ds = \oint_{n(c)} d\tilde{s} = 2\pi \left[\begin{matrix}\#\text{ of}\\ \text{times}\\ \text{around}\end{matrix}\right]$$

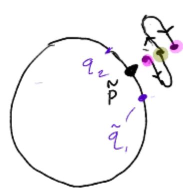
because c is a loop, $n(c)$ goes all the way around — no halvsies

In fact — the degree doesn't depend on $\tilde{p}$ so:

$$\oint_c \kappa\, ds = 2\pi \cdot \deg(n(c))$$

ex



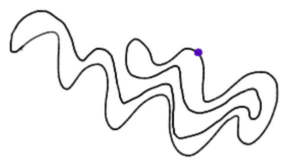


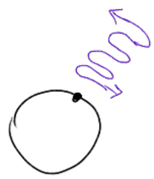
$\tilde{p}$ is covered 3 times, twice w/ positive curvature

once w/ negative

deg = 2 − 1 = 1

Hopf's Umlaufsatz : circulation theorem





- the "loop" forces the normal map to cover every point an odd # of times,
  - even $\Rightarrow$  something was missed
- the signs alternate $+, -, + -$, so $\deg(n(C)) = 1$

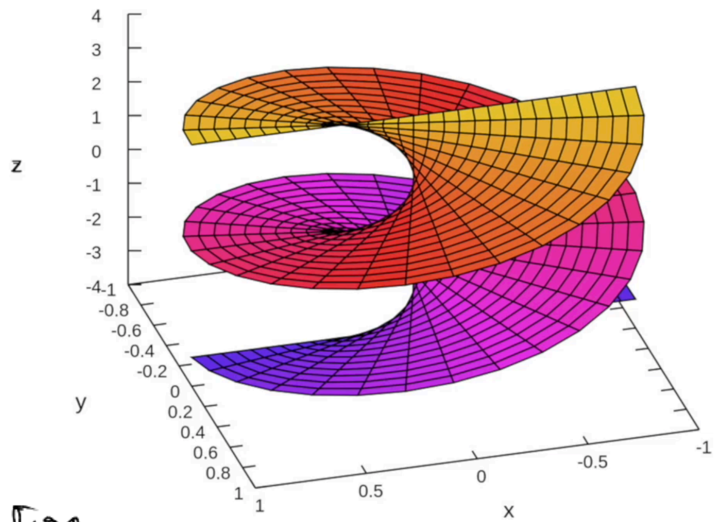
---

the surface case is essentially the same.

#19/

S_1: $f_1(r,\theta) = \ln r$ → 

S_2: $f_2(r,\theta) = \theta$



Compute K. If you know K @ every point, you can visualize the surface

Maybe helpful:

Curvature of surface of revolution

$K = -\ddot{y}/y$, p. 142

$$K = \frac{-1}{AB}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_u B}{A}\right)\right)$$

. Build metric on S_1 First