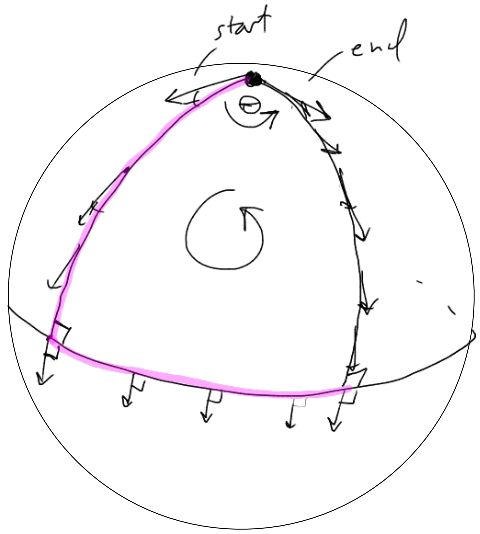
holonomy & Parallel Transport

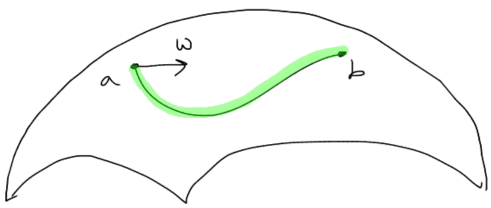


parallel transport along geodesics keeps the angle rel to path fixed. (length is preserved)

Counter-clockwise loop on sphere ⤳ counter-clockwise holonomy of θ.

Parallel Transport preserves angle wrt to initial vector

Intrinsic Parallel Transport



To P.T $\vec{w}$ along a non-geodesic curve $\mathcal{C}$, approximate $\mathcal{C}$ by a sequence of geodesic segments $\{G_i\}$ each of length less than $\epsilon > 0$, then carry $\vec{w}$ along G_i, maintaining constant angle w/ the direction of G_i. Take limit as $\epsilon \to 0$.

ex

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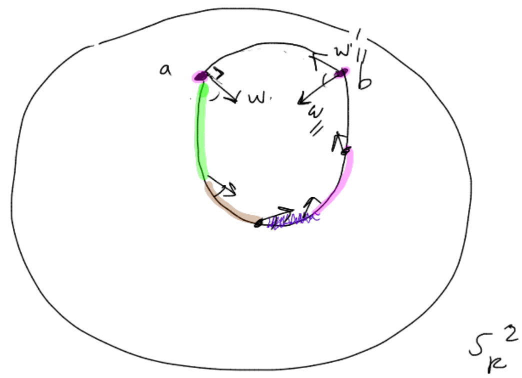
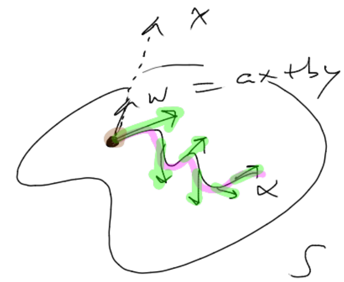


Holonomy:
Path dependent
in curved spaces

To measure how a vector changes wrt. parallel transport
we use the "covariant derivative."

$D_v w$ "intrinsic" co-variant derivative.

(In flat space, this is the usual directional derivative



For constants a, b,
f, g scalar functions
$\bar{v}, \bar{x}, \bar{y}, \bar{z}$ tangent to S

$$D_v(ax+by) = aD_v x + bD_v y$$

$$D_{fx+gy}(z) = D_{fx} z + D_{gy} z = f D_x z + g D_y z$$

product

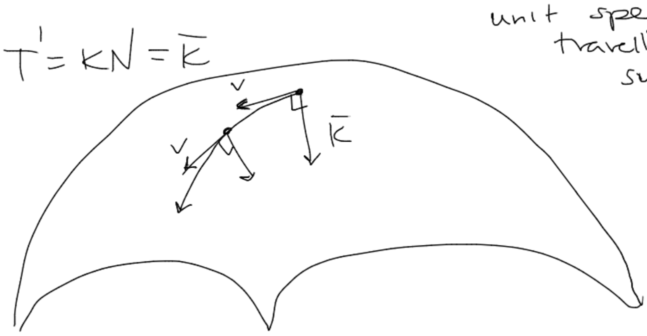
$$D_v(fx) = f \cdot D_v x + \overbrace{D_v f}^{f'}(x) = f D_v x + f' \cdot x$$

$$D_v(x \cdot y) = D_v(x) \cdot y + x D_v(y)$$

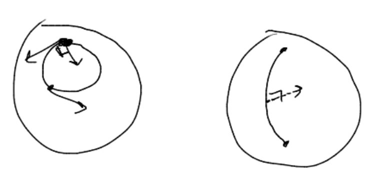
dot

$T' = \kappa N = \bar{\kappa}$

unit speed particle travelling along surface



ex



$$\bar{\kappa} = \bar{\kappa}_g + \bar{\kappa}_n$$

full acceleration

geodesic curvature

component of acceleration tangent to S

component of acc normal to S,

when $\bar{\kappa}_g = 0$ on geodesic

$$\iff D_v v = 0$$

[geodesic equation]

Holonomy

Midterm: Take-Home {Given: Wednesday?, Due: Next Wednesday} (mix exercises from text)
No class on Monday

the holonomy $R(L)$ of a simple closed loops L on surface S, is the net action on a tangent vector, when it is parallel transported about L.
(linear operator preserves length & angle)

(also, think holonomy as a Lin. Op acting on the entire tangent

* doesn't depend on starting / ending point

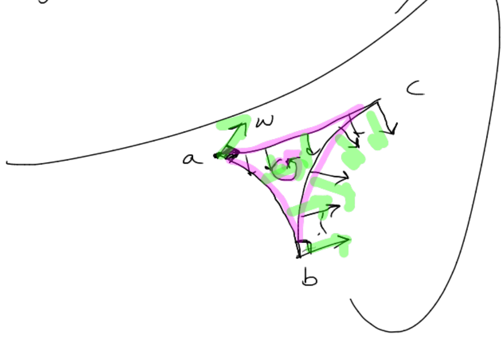
non-zero curvature $\longleftrightarrow$ non trivial holonomy action $\longleftrightarrow$ path dependence of parallel trans

zero curvature $\longleftrightarrow$ identity holonomy actio $\longleftrightarrow$ path indep. of parallel trans

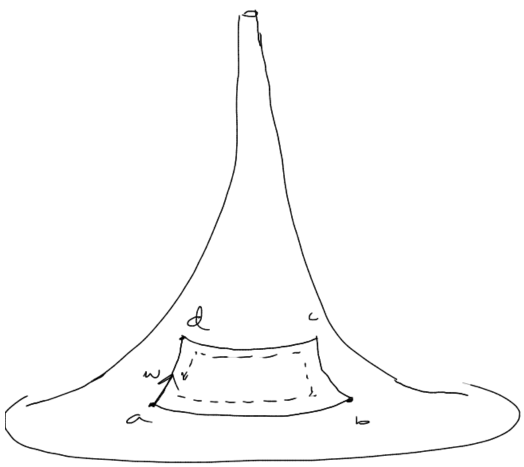
positive curvature $\longleftrightarrow$ direction of traverse of L (orientation of loop) is same as $R(L)$ rotation

negative curvature $\longleftrightarrow$ opposite.

Neg. Curvature Ex.



Note how parallel transported vectors are rotating clockwise.



conformally →