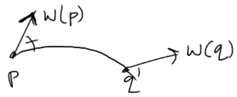
Parallel Transport Revisited

Eucl.

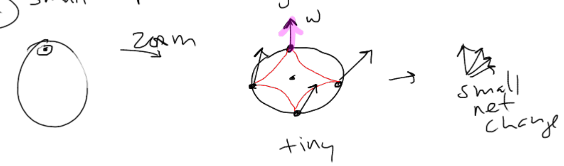

Sphere no $\parallel$ lines,
$\exists!$ geodesics for "most" pairs




ex

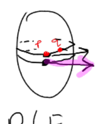
$S^2$ path-dependent holonomy

ex small loops vs big ones




big loop

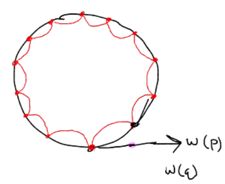
Equator = geodesic

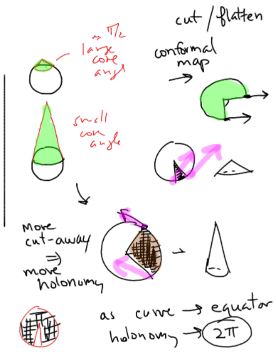


R(Equator) =

zoom

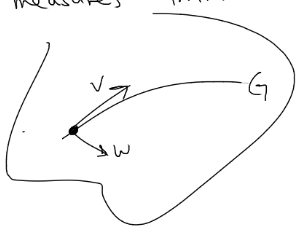
top view



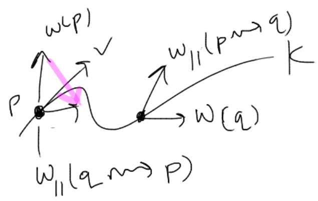
How is $D_v w$ defined? ($\nabla_v w$ = directional derivative)
Covariant derivative (intrinsic)
aka Levi-Civiti Connection

measures intrinsic rate of change of $w$ along $v$.




$v$ = velocity @ time $t$ of $G$ (tangent to $S$)
$w$ is also tangent to $S$

Def'n given $K \subset S$, velocity vector $v(t)$, $w$ tangent to $S$, defined everywhere along $K$,




$$D_v w = \frac{w_{\parallel}(q \rightsquigarrow p) - w(p)}{\varepsilon}$$

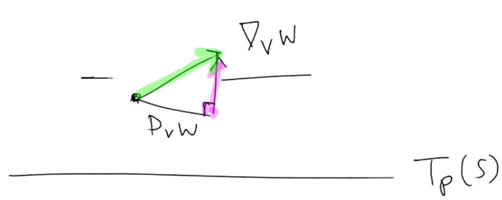
① properties

if $D_v w = 0 \overset{\text{def'n}}{=} \dfrac{w_{\parallel}(q \rightsquigarrow p) - w(p)}{\varepsilon} = w_{\parallel}(q \rightsquigarrow p) = w(p)$
i.e., $w$ is parallel transported along $v$

(ex) Find a non-geod. $K$ w/ vel vector $v$, and $w$ s.t. $D_v w = 0$.

② Recall: $\nabla_v w$ = rate of change of $w$ in direction of $v$.
$D_v w$ = rate of change of $w$ in direction of $v$ projected into the tangent space

$$= \rho(\nabla_v w) = \nabla_v w - (n \cdot \nabla_v w) \cdot n$$




directional derivative subtract its normal component

③ $D_v w = \nabla_v w - [w \cdot S(v)]n$ Show: $w \cdot S(v) = n \cdot \nabla_v w$

Recall. $S(v) = -\nabla_v n$
$w \cdot S(v) = w \cdot (-\nabla_v n) = \underbrace{-\, w \cdot \nabla_v n \overset{?}{=} n \cdot \nabla_v w}$

$0 = w \cdot \nabla_v n + n \cdot \nabla_v w$ product rule
$= \nabla_v (w \cdot n)$
$= \nabla_v (0)$

Connection b/w $\nabla_v w$ and geodesic curvature

$v \cdot (sv) = s \cdot v v$

Since $D_v w = \nabla_v w - (n \cdot \nabla_v w) \cdot n$

if $w = v$

$$D_v v = \nabla_v v - (n \cdot \nabla_v v) \cdot n = \bar{K}_g + \bar{K}_n - (n \cdot \underbrace{\nabla_v v}_{\bar{K}_g + \bar{K}_n}) \cdot n$$

$n$ = unit normal

$= \bar{K}_g + \bar{K}_n - (n \cdot (\bar{K}_g + \bar{K}_n)) \cdot n$

$= \bar{K}_g + \bar{K}_n - (\underbrace{n \cdot \bar{K}_g}_{=0} + n \bar{K}_n) \cdot n$

$= \bar{K}_g + \bar{K}_n - (\overbrace{n \bar{K}_n}^{K_n}) \cdot n \quad = K_g$

$(n \cdot K \cdot n) \cdot n$

$K (n \cdot n) \cdot n \quad = K_n$

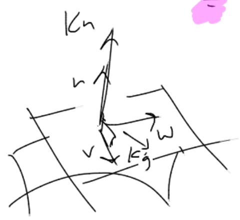
recall:
full accelleration

$\bar{K} = \nabla_v v$ has two component

$= \bar{K}_g + \bar{K}_n$ — normal component.

$\bar{K}_g$: geod. curv tan. to surface

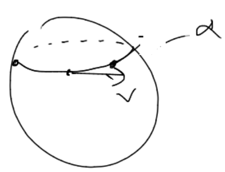
$\bar{K}_n = K_n$




$$\boxed{D_v v = K_g}$$

geodesic equation: when $D_v v = 0 \iff K_g = 0$

$K$ = geodesic




$D_v v \neq 0$.