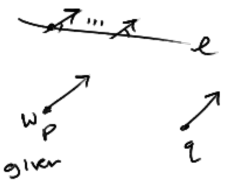
Parallel transport Revisited

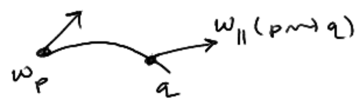
Eucl. Space



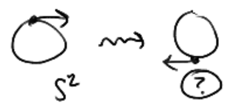
given w @ p, $w_{\parallel} = w_{p \rightsquigarrow q}$ is vector thru q $\parallel$ to p.
($\exists!$ direction)

Sphere : no $\parallel$ lines

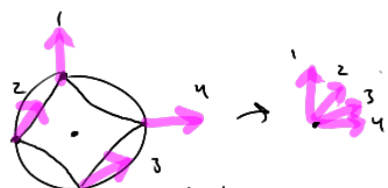
$\exists!$ geodesic segment b/w most points, use that



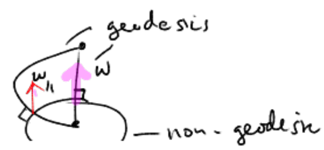
Q: what does p.transport look like for anti-podal pts?



Path Dependence!



small loop can be approx. w/ fewer geodesics



$\mathbb{H}^n$ ∞ $\parallel$ lines

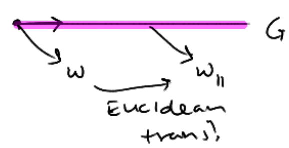
$\mathbb{H}^2$

How is $D_v w$ defined ?

Covariant (Intrinsic) Derivative, aka Levi-Civita Connections

measure's intrinsic rate of change of w along v.

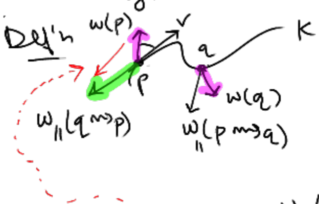
. If v = velocity of geodesic $\Rightarrow$ peel is flat:



$\Rightarrow$ replace peel

given curve $K \subset S$, w/ velocity $v(t)$, & w tangent to S, defined everywhere along K,

Def'n



$w_{\parallel}(q \rightsquigarrow p)$ is the parallel transport of $w(q)$, back to $w(p)$

$$D_v w = \frac{w_{\parallel}(q \rightsquigarrow p) - w(p)}{\epsilon}$$

properties:

(1) if $D_v w = 0 \Rightarrow \frac{w_{\parallel}(q \rightsquigarrow p) - w(p)}{\epsilon} = 0 \Rightarrow w_{\parallel}(q \rightsquigarrow p) = w(p)$

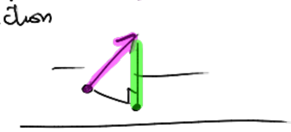
i.e., w is parallel transported along v

(2) $\nabla_v w \equiv$ rate of change of w in direction of v.

$D_v(w) \equiv$ rate of change of w in direction of v projected onto tangent plane

$$= \underset{\text{projection}}{P}(\nabla_v w) = \nabla_v w - (n \cdot \nabla_v w)\, n$$

"directional derivative subtract normal component of derivative."



(3) $D_v w = \nabla_v w - [w \cdot S(v)]\, n$

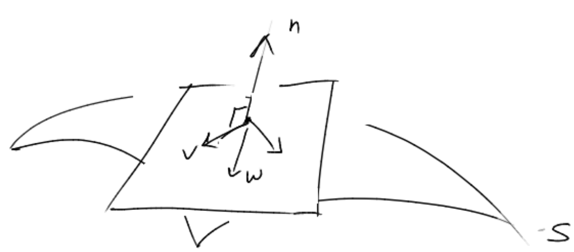
recall: $S(v) \equiv -\nabla_v n$

$$w \cdot S(v) = -w \cdot \nabla_v n \overset{?}{=} n \cdot \nabla_v w$$

$$0 \overset{?}{=} w \cdot \nabla_v n + n \cdot \nabla_v w$$

$$\overset{?}{=} w \cdot \nabla_v n + \nabla_v w \cdot n$$

$$\overset{?}{=} \nabla_v (w \cdot n)$$



w is tangent to surface i.e., $w \cdot n = 0$ so $\nabla_v (w \cdot n) = \nabla_v(0) = 0$

the transformations laws follow, eg,

since:

$$D_v w = \nabla_v w - (n \cdot \nabla_v w)\, n$$

$w = v \Rightarrow$

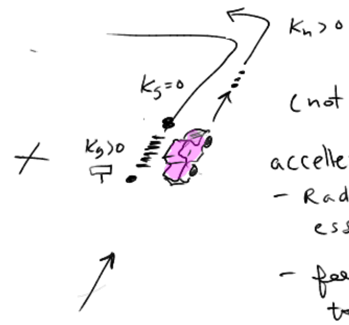
$$D_v v = \nabla_v v - (n \cdot \nabla_v v) \cdot n$$

recall: full accelleration

$\bar{K} = \nabla_v v$ has two components

$= \bar{K}_g + \bar{K}_n$ — component of acc. normal to S.

($\bar{K}_g$:) component of acc. tangent to S



(not unit speed param)

accellerate from stop sign

- Radius of Earth >> 0 so essentially flat
- feel acc. back, (parallel to) direction of travel

  $k_g > 0$

- curve, feel it ⊥ to direction

  $K_n > 0$

so

$$D_v v = \bar{K}_g + \bar{K}_n - \left[n \cdot (\bar{K}_g + K_n)\right] \cdot n$$

$$= \bar{K}_g + \bar{K}_n - [\underbrace{n \cdot \bar{K}_g}_{0} + n \bar{K}_n] \cdot n$$

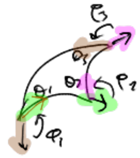
$$= \bar{K}_g + K_n - (n \cdot \bar{K}_n)\, n = \bar{K}_g$$

n = unit normal, $\bar{K}_n = k \cdot n$ so $(n \cdot k \cdot n) \cdot n = k \underbrace{(n \cdot n)}_{|n| = 1} \cdot n = kn = \bar{K}_n$

so $D_v v = 0 \Leftrightarrow \bar{K}_g = 0$ geodesic equation

Holonomy:

geodesic Δ:



$R(\Delta) = 2\pi - (\varphi_1 + \varphi_2 + \varphi_3)$ (Euclidean case: $R(\Delta) = 0$)

interior angle of n-gon ($n=3$)

Since $\varphi_1 = \pi - \theta_1$

$$R(\Delta) = 2\pi - \Sigma(\pi - \theta_i) = 2\pi - 2\pi + \underbrace{\Sigma\theta_i - \pi}_{\varepsilon(\Delta)} = \varepsilon(\Delta)$$

same for n-gon:

$$\boxed{R(P_n) = \varepsilon(P_n)}$$

& we'll see R is additive.

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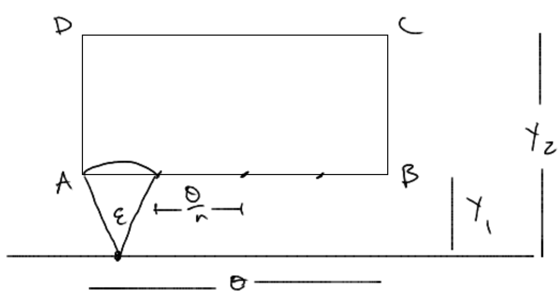
leads to

$$K(p) = \lim_{L_p \to p} \frac{R(L_p)}{A(L_p)} = \text{holonomy per unit area} \qquad \left(\approx \quad K(p) = \lim \frac{\varepsilon(\Delta p)}{A(\Delta p)} \right)$$

take $R=1$

ex

$\mathbb{H}^2$



$$A = \int_{x=0}^{x=\theta} \int_{y_1}^{y_2} \frac{dx\,dy}{y^2} = \theta\left(\frac{1}{y_1} - \frac{1}{y_2}\right)$$

$r\varepsilon \approx \frac{\theta}{n}$, $r \approx y_1$

$R_{AB} \approx -n\varepsilon \approx \frac{-\theta}{r} \approx \frac{-\theta}{y_1}$, $R_{CD} \approx \frac{\theta}{y_2}$

$$R = R_{AB} + R_{CD} = \frac{-\theta}{y_1} + \frac{\theta}{y_2} = -A \quad \Rightarrow \quad \underline{\text{Holon. per unit area} = -1 \text{ on } \mathbb{H}^2}$$

$\rightsquigarrow$ const. neg. curvature