# MA-163 (W,07)
## Test 5 (Series)

[150 total posible points. 140 possible EXTRA CREDIT points.] Name ____________________________

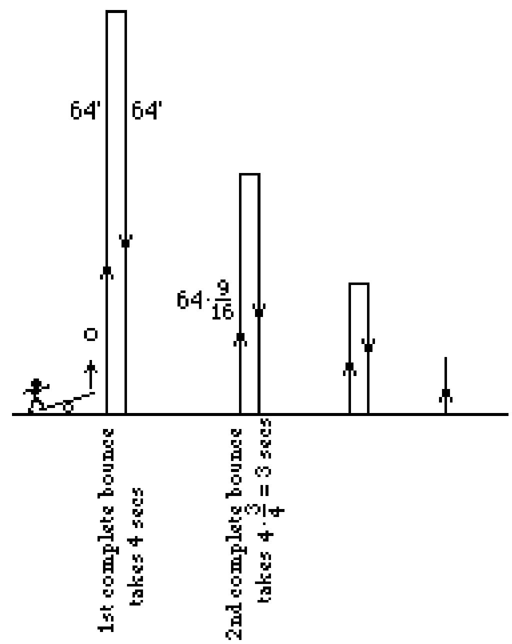
The instructions are on the bottom of page 4. Read them first.

---

[20 possible points]

1. A ball is catapulted from the ground with an initial velocity sufficient to have it reach the height of 64 feet. It falls back to the ground and then bounces up $\frac{9}{16}$'ths of its previous height; that is $\frac{9}{16}$ of 64 ft = 36 ft. It again falls to the ground and bounces up $\frac{9}{16}$'ths of the previous height from which it just fell. The ball keeps bouncing in this way until it finally comes to rest.

Also, 4 seconds are required for the ball to make its first complete bounce -- from the instant it leaves the ground until the instant it returns. And each subsequent complete bounce requires 3/4 times as long as the preceding complete bounce.




Answer (in sentences) these questions:

a. Write an infinite sum to give the total distant the ball travels in coming to rest. Then use an infinite series formula to find this distance.

b. Write an infinite sum to give the total time from the instant the ball is catapulted into the air until it finally comes to rest. Then use an infinite series formula to find this time.

---

[30 possible]

2 a. Use the definition of Maclaurin series to derive the series for $\frac{1}{1-x}$. [The definition of Maclauren series is on the formula sheet. Obviously, you must show your work in detail since the series for $\frac{1}{1-x}$ is also given on the formula sheet.]

b. Use the series you derived in (a) to obtain a series for $f(x) = \frac{1}{1+x^2}$. [If you didn't get (a), you can still do (b) and (c), using the series on the formula sheet.]

c. Then integrate the latter series to obtain a series for arctan x. [If you're uncertain, I'll correct your answer for (b) [or give you the correct answer] before you do (c).]

---

[20 possible]

3. Suppose the function f(x) can be expressed as a Maclauren series by

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{3^{n-1}}{n} \cdot x^{n-1} = 1 - \frac{3}{2}x + \frac{9}{3}x^2 - \frac{27}{4}x^3 + \dots \ .$$

Use the Ratio Test to determine the interval of convergence. Be sure to check the endpoints. Show all your work. Include explanations and reasons in your answer. Your work and explanations will be graded.

---

[20 possible]
4. The Maclauren series for $f(x) = \sin x$ is given on the formula sheet. Use the series for $\sin x$ to obtain a series $\sin x^2$. Simplify your result as much as possible. Then write the series in summation notation. [You must show your work so I can see how you're getting the result. Also, if you're uncertain about your series, ask me to check your answer now for a few points off. If you have it wrong, I'll give you the correct series so you can do the next few problems.]

---

[20 possible]
5. Let $P_1(x)$ and $P_2(x)$ be the first two terms in the sequence of partial sums of the series for $\sin x^2$. (These are just the first two Taylor polynomials for the function.)

a. Write the equations for $P_1(x)$ and $P_2(x)$. That is, complete $P_1(x) =$ ________ and $P_2(x) =$ ________, but do it on your answer sheet. [Again, if you're uncertain about your answer, ask me to check your answer now for a few points off. If you have it wrong, I'll give you the correct answer so you can do (b).]

b. On an accompanying page of this test, I've drawn the graaph of $y = \sin x^2$. On the same coordinate system, draw **large, accurate** graphs $y = P_1(x)$, and $y = P_2(x)$. For both, use as much of the coordinate system as you can. You may use your grapher but you must reproduce the graphs on your paper. Tell which graph is which.

---

[20 possible]

6. Use your series (**not** NUMINT) for $\sin x^2$ to approximate $\int_0^{5/4} \sin x^2 \, dx$ correct to three decimal places. [Show your work. You may check your answer with NUMINT if you wish.]

---

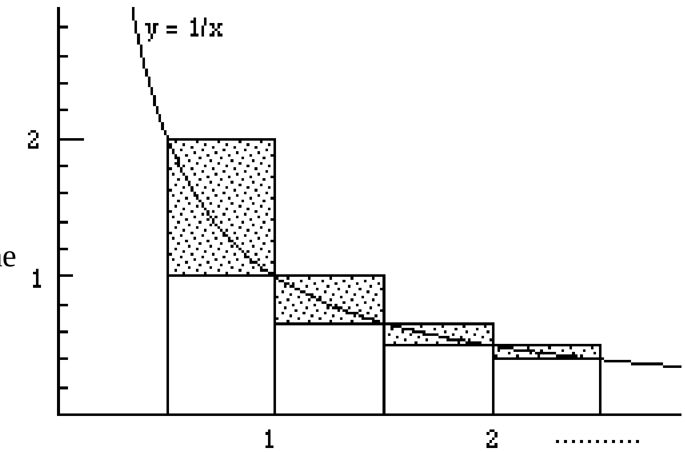
[20 possible]
7. Use your series for $\sin x$ [Note: That's $\sin x$, not $\sin x^2$.] to obtain a series for $\cos x$. Show what you're doing to get your result. [If you can't do this problem, you can do EC Problem 11 instead.]

*******************************************************************************

The remaining problems on the next two pages are for EXTRA CREDIT. Do them only after you've completed, and checked, your solutions to Problems 1 - 7.

[14 possible]
8. (a) By writing out the series, find the total area of the shaded region at right, assuming the pattern continues forever.

(b) Suppose the shaded region above was revolved about the x-axis. The result would be a kind of a cone-shaped, stair-stepped solid. What is the volume of the solid?




---

[30 possible]
9. There are a number of tests to determine when series converge and when they diverge. The **Integral Test** is one of these. It says

**Integral Test.** Suppose you have a series $\sum_{n=1}^{\infty} a_n$ . If there is a function, f, that is continuous, positive and decreasing on $[1, \infty)$ and $f(n) = a_n$ , then (i) if $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent; and (ii) if $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Here's your question: Use the Integral test to determine the convergence or divergence of these three series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ; (b) $\sum_{n=1}^{\infty} \frac{1}{n}$ ; (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ .

---

[16 possible]
10. The point of this question is to see if you understand the terminology and the concepts related to sequences and series. [You may put your responses in the blanks below.]

Put T (true), F (false), or M (meaningless) in front of each of these statements:

_____a. The limit of the sequence .3, .33, .333, ... is 1/3.

_____b. The limit of .333... is 1/3.

_____c. Given the series $\sum_{n=1}^{\infty} a_n$. If the sequence $a_1, a_2, a_3, \ldots$ has limit zero, then the series has a sum.

_____d. Given the series $\sum_{n=1}^{\infty} a_n$. If the sequence of partial sums of the series $A_1, A_2, A_3, \ldots$ has limit zero, then the series has a sum.

_____e. Given the series $\sum_{n=1}^{\infty} a_n$. If the sequence $a_1, a_2, a_3, \ldots$ has limit zero, then the sequence of partial sums of the series has a limit.

_____f. Even though $\frac{1}{1-3}$ is defined, and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$, the number 3 is not in the domain of the function f defined by $f(x) = 1 + x + x^2 + x^3 + \ldots$.

_____g. The terms of the sequence .9, .99, .999, ... approach, but never reach, the number 1.

_____h. .999... approaches, but never reaches, the number 1.

---

[30 possible]

11. The Maclauren series for $e^x$ is given on the formula sheet. Use it to obtain a series for $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

Then use your result to approximate $\int_0^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$.

---

[30 possible]

12. First, find a power series representation for $f(x) = \ln(1 + 2x)$. Then find a power series representation for $g(x) = \ln(1 + 2x)^{1/x}$. Next use the series for g to find $\lim_{x \to 0} g(x)$. Finally, use your previous result to find $\lim_{x \to 0} (1 + 2x)^{1/x}$.

---

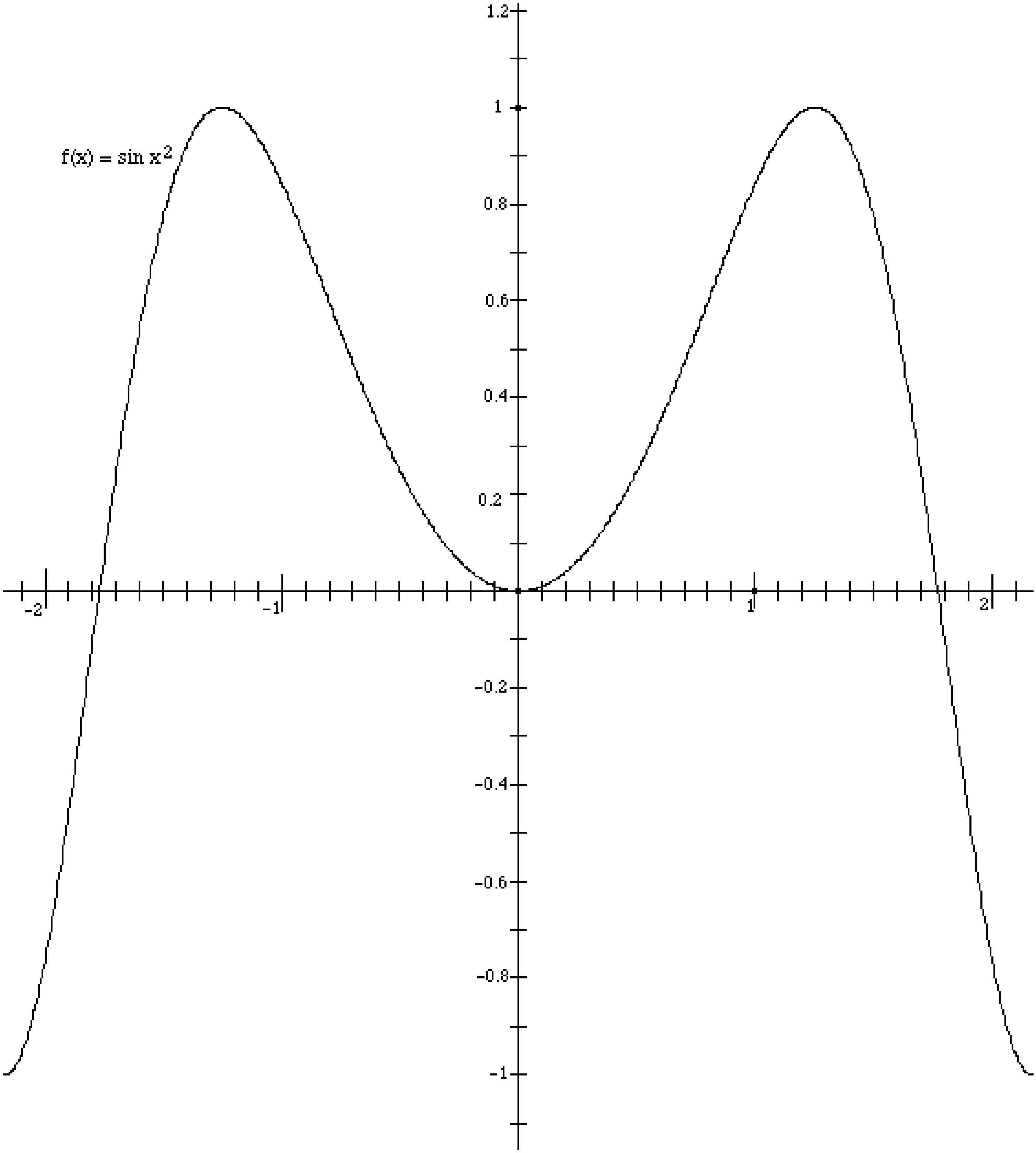
[20 possible]

12. You have series for cos x (on the formula sheet) and for sin x (in Problem 6.) Use these two series to find a series for tan x.

*****************************************************************************

INSTRUCTIONS:

Do the problems on the paper provided, numbering the problems clearly. Be sure your name is on each paper. Your work will be graded. **Neatness, notation and logic all count.** For answers that involve series, you do not have to use summation notation unless you are explicirtly told to do so or unless you need to do so to use a convergence test. However, you do have to simplify the terms of the series and include enough terms (usually five or six), so that I can tell you see the pattern.


f(x) = sin x2
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-2
-1
1
2