MA-161 (F,06)
Test 2 (Derivatives)

[160 possible points. Each problem is worth 20 points.] Name______________________________

DIRECTIONS and FORMULAS: On another sheet. This test is based on the summary of the material you studied in this Unit. The summary was given yesterday in class.
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1. The starting point was the definition of derivative. The definition was used to derive formulas for the derivatives of four elementary functions, namely (i) For any natural number n, $\frac{d}{dx}(x^n) = n\cdot x^{n-1}$; (ii) $\frac{d}{dx}(\sin x)$; (iii) $\frac{d}{dx}(\cos x)$; and (iv) $\frac{d}{dx}(e^x)$. Choose <u>one</u> of these formulas and derive it, <u>using the the definition of derivative</u>. Formulas (i) and (iii) are worth full credit, formula (ii) is worth 90% credit, and formula (iv) is worth 80% credit. You must show all your steps. The special limits on the Information Sheet would be helpful. [Note: If you can't do any of (i) - (iv), you may derive the formula for $\frac{d}{dx}(x^2)$ for 70% credit.]

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2. The definition of derivative was also used to derive formulas for the derivatives of combinations of functions, the first three of which were
(i) The Constant Multiple Rule; (ii) The Sum Rule: $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$; and
(iii) The Difference Rule: $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$.
<u>Use the definition of derivative</u> to derive The Sum Rule. [Note: If you can't do The Sum Rule, you may state and derive The Constant Multiple Rule for 85% credit. A correct <u>symbolic</u> statement of the rule is itself worth 3 points.]

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3. The <u>Chain Rule</u> turned out to be very useful in finding derivatives of functions that could be expressed as the composition of functions. Use the Chain Rule to do the following problems:
(a) Find $\frac{dy}{dx}$ if $y = \sin^n x$. (b) Find $\frac{d}{dx}(\sin x^n)$. (c) What is $D_t\left(e^{5t^3}\right)$? (d) If y is a function of x, what is $D_x\left(e^y\right)$?

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4. After the Chain Rule, you learned about Implicit Differentiation and, in particular, about the Derivative of an Inverse Function Procedure. [An example of this procedure is given on the Information Sheet.] Using this Procedure, it is possible to derive the formula for the derivative of the natural log function, namely $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Show how this formula is derived, <u>using the Derivative of an Inverse Function Procedure</u>. Your derivation should begin "Let y = ln x. Then ....". [Note: If you can't do $\frac{d}{dx}(\ln x)$, you may use the derivative of an inverse function procedure to derive the formula for the derivative of arccos x; that is, derive the formula for $D_x(\cos^{-1} x)$. This would be for 90% credit.]

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5. Once you have the formula for $\frac{d}{dx}(\ln x)$, it is possible to use <u>Logarithmic Differentiation</u> to derive many other formulas. [An example of Logarithmic Differentiation is given on the Information Sheet.] I showed you in class how logarithmic differentiation could be used to derive the Quotient Rule. You derive the Product Rule by the same method. Your derivation would begin "Let $y = f(x)\cdot g(x)$. Then $\ln y = \ln(f(x)\cdot g(x))$. ...." [Note: If you can't do the Product Rule, you may, for 90% credit, <u>use Logarithmic Differentiation</u> to derive the formula
$\frac{d}{dx}(x^r) = r\cdot x^{r-1}$ for any real number r.]

6. With all of these formulas and methods in hand, it is possible to differentiate any function that is given as a combination of the elementary functions. (That's what you were doing on all the derivative rules quizzes.)

Find the derivative with respect to x for each of these functions. a is a positive constant. Simplify your result. <u>Do not forget to use the Chain Rule when appropriate.</u>

(a) $y = 2 - \frac{1}{6}x^3$; (b) $g(x) = \sqrt{a^2 - x^2}$; (c) $y = x \cdot \ln x - x$; (d) $f(x) = \tan^{-1}(x^2)$; (e) $y = x^{\sin x}$.

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7. As usual, we make things more complicated by building new concepts on old. In particular, you were introduced to the idea of higher order derivatives. The second derivative of a function is the derivative of the first derivative; i.e., $\frac{d^2}{dx^2}\big(f(x)\big) = D_x^2\big(f(x)\big) = y'' = \frac{d}{dx}\left(\frac{d}{dx}\big(f(x)\big)\right)$. Higher derivatives are defined in the same way. (Don't forget the Chain Rule in these problems!) (a) If $y = e^{\frac{1}{2}x^2}$, find both (i) y" and (ii) $\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{2}}$.

(b) Find $\frac{d^3}{dx^3}\big(e^{2x}\big)$. (c) Then find a formula for $D_x^n\big(e^{2x}\big)$. Show where your formula comes from.

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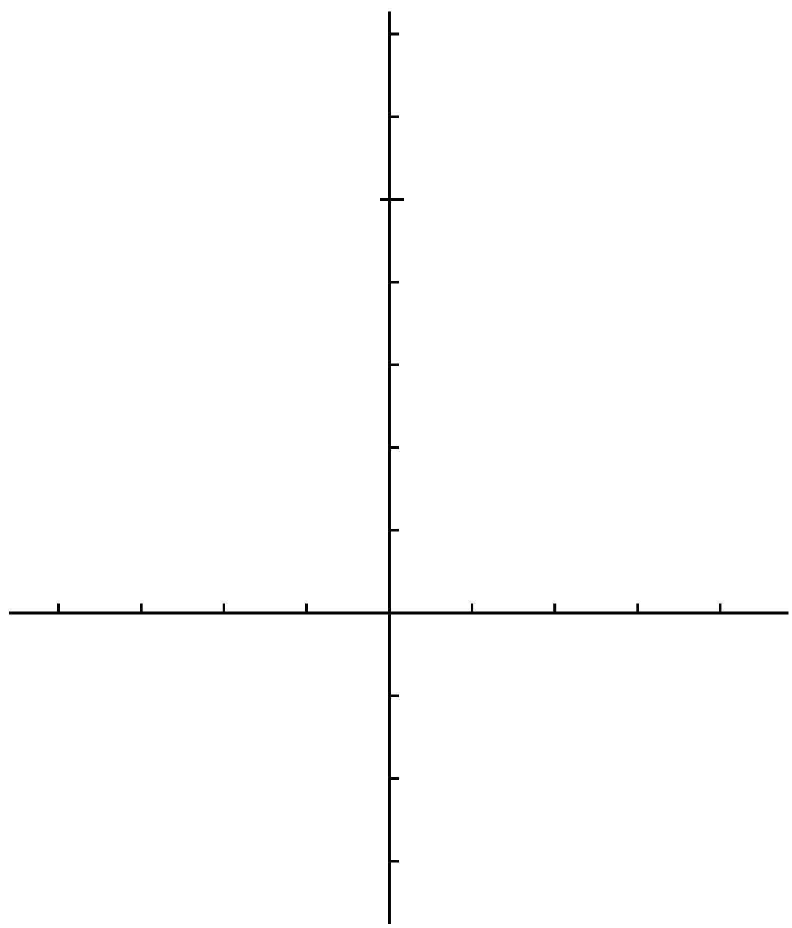
8. The major application of derivative we focused on in this Unit was its use to find the slope of a tangent line to a function at a given point. On the coordinate system I've provided below, draw a reasonably accurate graph of $y = 2 - \frac{1}{6}x^3$ for $-3 \le x \le 3$. Find the **exact** (<u>not</u> estimated) slope of the tangent line to the graph at the point where x = 1. Give the (**exact**) equation of this tangent line. Finally, draw carefully this tangent line on your graph. (You may do all your work, and give your answers, in the space below.)



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The following problems are for EXTRA CREDIT. However, do not work on these problems until you have done all you can do, and have checked, problems 1 – 8. Each EC problem is worth 20 points.

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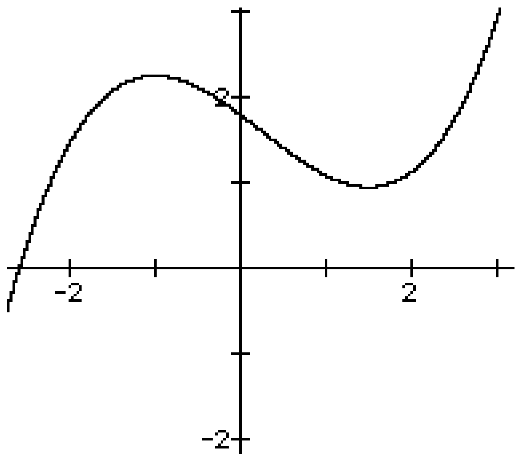
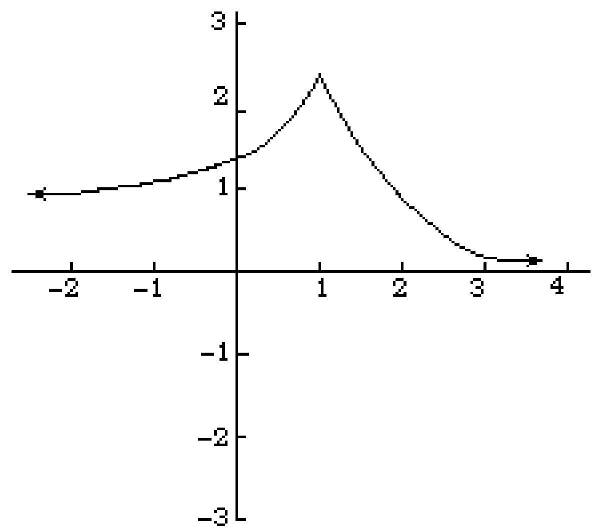
9. Below are the graphs of two functions. On the same coordinate systems, draw approximate graphs of the derivatives of the these functions. Be sure your sketches are consistent with the important features of the original functions.





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10. Suppose your roommate, Pat, is taking precalculus so knows about slopes of lines and about graphs of the trigonometric functions. Pat also knows that the derivative of a function is the slope of the tangent to the graph of the function. Draw pictures and explain how you could make it plausible to Pat that the derivative of the sine function is the cosine function.

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11. Differentiate each of the functions $y = (x^3 + 1)^5$ and $y = 5^{(x^3+1)}$

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12. Find the equation of the line tangent to the graph of $y = e^{-\frac{1}{2}x^2}$ at the point $(1, 1/\sqrt{e})$.

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13. Differentiate and simplify: (k is a positive constant.)

(a) $D_x\left(\frac{1}{k}\arctan\frac{x}{k}\right)$ (b) $D_x\left(\frac{1}{2k}\ln\left(\frac{x-k}{x+k}\right)\right)$ (c) $t(x) = \frac{1}{n^2}x^n(n\ln x - 1)$ where n is a natural number.

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14. There are two lines that are tangent to the curve $y = \frac{x}{x-1}$ which pass through the point (2, 0). At which points do these lines touch the curve?

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15. A function f has derivative $f'(x) = 3x^2 - 4x + 1$ and its graph contains the point (2, 6). What is the equation of the function f?
