MA-161 (F,06)
Test 3
Applications of the Derivative

[155 possible points] Name ________________________________

The instructions are on another sheet. **Read them first**.

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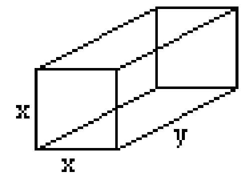
[35 possible]

1. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length. The total length of all the wire used to make the frame is L feet where L is a positive constant. The ultimate questions are these: What are the dimensions of the frame that makes a box with the largest possible volume? What is the volume in this case?

Let x be the lengths of the sides of the square ends of the frame and y the lengths of the connecting wires. Let V be the volume of the resulting box.



a. Using the fact that the total length of wire is L feet, write y as a function of x. [The constant L will appear in your formula as well.]

b. Write the volume of the box, V, as a function of x. **Simplify** the formula as much as possible before doing (c).

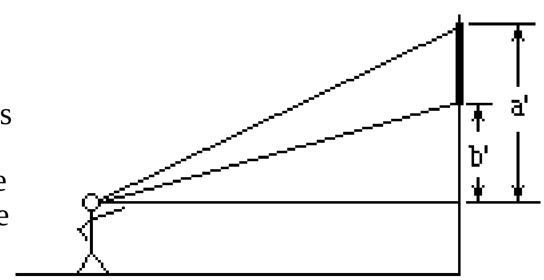
c. Use the methods of calculus to find the value of x that makes V a maximum. [Remember that L is a constant and the derivative of a constant is zero!] Use the Second Derivative Test to show that the value you got does maximize the volume. [Show all your work so I can tell exactly what you're doing. Show explicitly your use of the Second Derivative Test.]

d. Answer the two questions in the original problem. [Don't forget units of measurement.]

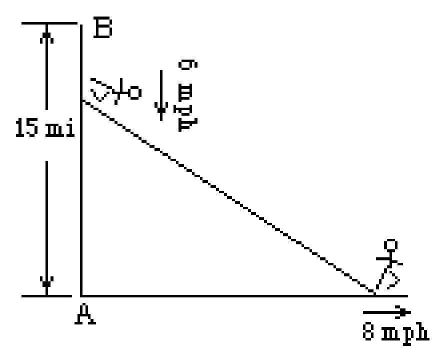
[35 possible]

2. Choose one of these two problems. The first contains some parameters so is worth full credit. The second is less general so is worth 90% credit. If you can't do either of these, you can do EXTRA CREDIT Problem 7 for 80% credit.

A. The top of the window in a prison cell is a feet above the resident prisoner’s eye level. The bottom of the window is b feet above his eye level. Since the prisoner knows calculus, and he has plenty of time on his hands, he decides to determine how far he should stand from the wall in order to maximize his viewing angle of the window. Where should he stand in order to make the angle as large as possible?



b. Town B is 15 miles directly north of Town A. Alice leaves Town A at noon running directly east at a constant speed of 8 miles per hour. Also at noon, Betty leaves Town B running directly south toward Town A at the constant speed of 6 miles per hour. At what time will the two women be nearest to one-another?



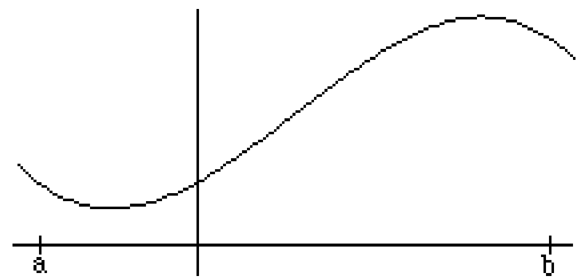
[30 possible]

3. **No graphing calculators on this problem.** Use the methods of calculus to answer this question.

Let $f(x) = 3x^2 - x^3 - 1$. Do a graphing analysis for this function. [I.e., Find the relative extrema and inflection points. Use the Second Derivative Test where relevant, stating what your conclusion is. Tell the intervals on which f is increasing and on which it is decreasing, and the intervals where the graph is concave up and where it's concave down. Draw the graph, labeling the extrema and inflection points on the graph. **On the coordinate system I've provided**, draw the graph of f, labeling the extrema and inflection points on the graph. Write the graphing summary on that page as well. Your graph should be at least for $-1 \le x \le 3$.

[25 possible: 18 for a; 7 for b]

4a. Complete this statement of the Mean Value Theorem (MVT) in the space below. Draw lines in the figure at right to illustrate its geometric meaning, labeling the appropriate points. [You may complete the statement and draw the figure on this paper.]



MVT) Given a function f which is defined on the closed interval [a, b]. If ...

b. Use the MVT to prove the following theorem: If the function f is differentiable on [r, s] and $f'(x) > 0$ for all values of x in [r, s], then $f(r) < f(s)$.

Note: If you can't do (b), you may do instead EXTRA CREDIT Problem 8.

[30 possible]

5. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 5/2 feet per second.

a. How fast is the area enclosed by the ripple changing when the radius is 8 feet?

b. How fast is the area changing when the area is 36 square feet?

c. How fast is the circumference of the circle changing in 2 seconds?

Note: If you can't do this problem, you may do EXTRA CREDIT problem 9 for 80% credit.

[30 possible: 13 for a; 12 for b]

6a. Draw a small portion of the graph of $y = f(x)$ near the point $(a, f(a))$ in the first quadrant subject to these conditions: $f'(a) = 0$ and $f''(a) > 0$. You must include the tangent line in your graph.

b. Draw a portion of the graph of $y = f(x)$ near the point $(2, f(2))$ given that f satisfies these conditions: $f(2) = 3$, $f'(2) > 0$, $f''(2) = 0$, $f''(x) < 0$ for $x < 2$, and $f''(x) > 0$ for $x > 2$. You must include the tangent line in your graph.

The following problems are for EXTRA CREDIT. Do not work on these problems until you have done all you can do, and have checked over, problems 1 – 6.
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7. Suppose $a_1, a_2, ..., a_n$ are constants, and the function f is defined by the equation
$f(x) = (x - a_1)^2 + (x - a_2)^2 + ... + (x - a_n)^2$. Show that the value of x that makes f a minimum is when x is the average of the a_i's. That is, show that f has a minimum at $\bar{x} = \frac{a_1 + a_2 + \ldots a_n}{n}$. [Do this by using the usual method for determining the value of x that minimizes f.]

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8. First, draw the graph of $f(x) = x^2$. Then choose two general points on the graph -- point A with coordinates (a, f(a)) and point B with coordinates (b, f(b)). Draw the secant line $\overleftrightarrow{AB}$ and find its slope. Simplify your result. [Remember, these are points on the graph of $f(x) = x^2$.] Next, using f', find the number c in the interval (a,b) that satisfies the conclusion of the MVT. Your answer will involve the numbers a and b.

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9. Assume that a spherical snowball melts in such a way that its radius decreases at a constant rate. Suppose it begins as a sphere of radius 12 cm and it takes 3 hours to disappear. At what rate is the volume changing in one hour? Be sure to define variables and answer in a sentence with appropriate units of measurement. [Reminder: The volume of a sphere is $(4/3)\cdot\pi\cdot(\text{radius})^3$.]

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10. What are the dimensions of the rectangle with largest area that can be inscribed in the ellipse with equation $\frac{x^2}{r^2} + \frac{y^2}{s^2} = 1$.

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11. Think about a cylindrical propane tank with hemispherical ends like the one shown at right. You want to construct such a tank to have a volume of 500 cubic feet. However, you want the surface area to be as small as possible. What should the dimensions of the tank be to give the smallest possible surface area? What would the tank look like?

Note: Volume of sphere = $(4/3)\pi(\text{radius})^3$ Surface area of sphere = $4\pi(\text{radius})^2$

Volume of cylinder = $\pi(\text{radius})^2\cdot\text{height}$ Lateral surface area of cylinder = $2\pi(\text{radius})\text{height}$

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12. Sketch a graph of $f(x) = e^{-\frac{1}{2}x^2}$. Let w be half the width of an arbitrary rectangle with one side on the x-axis and with the vertices of the opposite side on the graph of f. Let A(w) be the area of the rectangle. Show that the one of these rectangles with largest possible area has its upper vertices on the inflection points of the graph of f.