MA-331 (W,07)
Test 1 (Unit 1)

[250 possible points] Name ________________________

The instructions are on the bottom of page 3. Read them first before you begin.

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[30 possible points]

1. For the structure < **F**, +, · > to be a field, twelve axioms must be satisfied. Here are eleven of the axioms: **F** must be closed under + and · ; the associative and commutative properties must hold for + and · ; **F** must contain additive and multiplicative identities, say 0 and 1; every element of **F** must have a unique additive inverse in **F**; the distributive property must hold; and the two identities must be different.

(a) Write in symbols precisely what it means for a set S to be closed under the operation of .

(b) Write precisely in symbolic form the remaining field axiom. (Do not just give its name.)

[40 possible]

2. We began with a set, **R** (whose elements were called real numbers), together with two binary operations, + and ·, defined on **R**. We assumed that < **R**, +, · > was a field. Using just the field axioms and some definitions, we were able to prove a number of theorems that would have to hold in any field.

Using just the axioms, definitions and theorems proved in class or on homework assignments, give a careful proof of this theorem:

a, b, x **R**: If a is not 0 and $\frac{x}{a} - b = 0$, then $x = ab$.

Be careful about grouping symbols and about the use of the definitions of subtraction [$x - y = x + (-y)$] and division [$\frac{x}{y} = xy^{-1}$]. Also, if you're in doubt about whether or not something you want to use has been previously proved, you should either ask me or prove it here or, best, just avoid using it.

[10 poss]

3. To be an ordered field, there must be a relation, <, defined on **F** which satisfies four axioms. They are Trichotomy, Transitivity, Additivity (Additivity says a,b,c **F**: if a < b, then a + c < b + c) and Multiplicativity (Multiplicativity says a,b **F**: if 0 < a and 0 < b, then 0 < ab.)

What, precisely, does the Trichotomy Axiom say? [If you don't know, you may state the Transitive Axiom for less credit.]

[40 possible]

4. We next assumed that < **R**, +, ·, < > was an ordered field.

Using just the order axioms and the field axioms, definitions and theorems proved in class or on homework assignments, give a careful proof of these two theorems:

Th 1. x **R** : x < 0 iff 0 < –x. Th 2. y,z **R** : if 0 < y and z < 0, then yz < 0.

To prove these, you may use any of the field theorems we proved in class but you may not use any of the theorems about < that were proved in class. However, you may use Theorem 1 in your proof of Theorem 2.

[30 poss]

5. [Remember: **Q** is the set of rational numbers and **N** is the set of natural numbers.] Define the sets A, B and C as follows: A = { x | x **Q** and x ≤ 2 }, B = { x | x **Q** and $x^2 \le 2$ },

and C = { x | x **R** and $x = 2 - \frac{1}{n}$ for some n **N** }.

Answer these question about the three sets: [Before doing so, you might first try to determine what the elements in these sets look like.]

a. Show that set B is non-empty and is bounded above.

b. What is the least upper bound of A? of B? of C?

c. Is the least upper bound of A an element of A? of **Q**? of **R**?

d. Is the least upper bound of B an element of B? of **Q**? of **R**?

[30 poss]
6. The Completeness Axiom says

If S is any non-empty subset of **R** which has an upper bound, then S has a least upper bound (lub) in **R**.

(a) What precisely does it mean to say that the number b is an upper bound of S? ["Precisely" means to give a good mathematical definition, so your definition would begin "Let S be a subset of **R**. The number b is ... "]

(b) What precisely does it mean to say that b is a least upper bound of S?

(c) We know that the Completeness Axiom holds for the set of real numbers because we have postulated that it holds. Does the Completeness Axiom hold for **Q**? That is, if "**R**" is replaced by "**Q**" in the statement above, will the statement be true? Give an arument to justify your answer.

[40 poss]
7. Let **F** = { 0, 1, 2, 3, 4, 5, 6 } with + and · defined on **F** by the tables at right. (These are just the integers mod 7.) It is a fact – which you do not have to verify – that this structure is a field. Answer these questions about this field.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

[Note: All of your numerical answers to the questions below, including g, h, and i will be 0, 1, 2, ..., 6 or "No such number exists in F."]

The additive identity for this field is 0 because x + 0 = x for all x in **F**. Similarly, the multiplicative identity is 1.

a. What is the additive inverse of 3; i.e., –3 = ? Explain why.

b. What is the multiplicative inverse of 5; i.e., 5^{-1} = ? Explain. c. $\frac{3}{5}$ = ?

d. What is the solution for the equation $\frac{x}{5} - 2 = 0$? Show that your solution is the same as the theorem you proved in problem 3a says it should be.

e. Solve: $x^2 = 5$. f. Solve: $x^2 = 2$.

The following questions -- g, h, and i -- still refer to the field, F, above.

g. Define **C** to be the smallest subset of **F** = {0 ,1, 2, 3, 4, 5, 6} that contains 1 and is closed under adding 1. List all the elements of **C**.

h. Define the set **I** as follows: x **I** iff x **C** or x = 0 or x = –n for some n in **C.** List the elements of **I**.

i. Define **L** by **L** = { x | x = pq^{-1} for some p and q ≠ 0 in **I** }. List the elements of **L**.

[I'm sure you've noticed that the sets **C**, **I**, and **L** are defined in precisely the same way the natural (i.e., **C**ounting) numbers, the **I**ntegers, and the Rationa**L** numbers were defined when we did the real numbers.]

[30 poss]
8. In developing the theory of the real numbers, our initial goal was to state a set of axioms that would permit us to prove the things we know to be true about the real numbers. We did this by first assuming that the real numbers formed a field, then that they formed an ordered field, and finally that they formed a complete ordered field. As we stated the axioms, we continued to ask ourselves whether or not particular assertions could be proved using just the axioms we had assumed up to that point. (**Note:** For both (a) and (b), it is not the answer to the question that's important; instead, ii's the detailed explanation that will be graded.

a. Is it possible to prove the statement below, assuming that $\langle$ **R**, +, ·, < $\rangle$ satisfy just the axioms for an ordered field? Explain in detail how you know. The statement is

a R: If 0 < a, then x R such that x^2 = a.

b. If we assumed that $\langle$ **R**, +, · $\rangle$ satisfied just the axioms for a field, could we prove that some integers are not natural numbers. That is, using just the field axioms, could we prove the assertion

There is at least one element in the set of integers which is not in the set of natural numbers.

Explain how you know. (You might want to look at Problem 7 above.)

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EXTRA CREDIT. Do this only after you've completed everything else you can do on the test. Each problem is worth 50 EC points. You may tear off this page and do these problems at home for less credit. You must do them on your own.

9. Answer these questions:

a. Do the integers form a field? Explain.

b. Do the mod five integers (with + and · as given in Problem 4) form an ordered field? Explain.

c. Do the rational numbers form an ordered field?

d. Do the rational numbers form a complete ordered field? Explain in detail how you know.

e. Do the real numbers form a complete ordered field?

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10. Irrational numbers are defined as follows:

x is irrational iff x is a real number but x is not rational.

We've assumed that the real number system is a complete ordered field. The Completeness Postulate implies that the least upper bound of the set

$S = \{ x \mid x \text{ is a rational number and } x^2 < 2 \}$

is a real number.

a. Explain how this guarantees that there is at least one irrational number in the set of reals.

b. Once you know that there is at least one irrational number, it is possible to prove that there are a lot more of them. Restate and prove this theorem, using an **indirect proof**: [You will have to restate the theorem in symbols.]

The sum of a rational number and an irrational number is irrational.

c. Is the product of an irrational and an irrational itself irrational? Explain.

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11. Prove that, in any field, if the number a satisfies the equation $x^2 = k$ (where k is an element of the field), then so does the additive inverse of a. (You may use any of the field theorems that were proved early in the course.)

Then show that this is true for the field $\langle \mathbf{F}, +, \cdot \rangle$ given in Problem 7 by looking at the equations $x^2 = 1$, $x^2 = 2$ and $x^2 = 4$ in that field.

[Remember that $\langle \mathbf{F}, +, \cdot \rangle$ is the integers mod 7 so $\mathbf{F} = \{0, 1, 2, 3, 4, 5, 6\}$.]

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Instructions: Unless otherwise indicated, write all your answers on your answer sheets. Be sure to number the pages and the problems. All answers must be given as complete **sentences** unless otherwise specified. [A sentence in mathematics could look like this: $2^{-1} = 7$.] Be careful about your use of notation. Do **not** circle or box "answers."