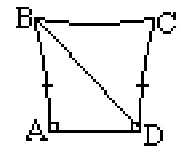
MA-331 (W,07)
Test 3

[200 possible points; 50 possible EC points] Name ________________________

The instructions are on thebottom of page 2. Read them first.

[40 possible: 10, 10, 20]

1a. I've drawn a picture of a Saccheri quadrilateral at right. We proved the Saccheri Theorem in Absolute Geometry. On your answer sheet, tell what the Saccheri Theorem says about the relationship between the upper base and lower base in terms of the figure. Does this relationship also hold in Euclidean Geometry? Explain.

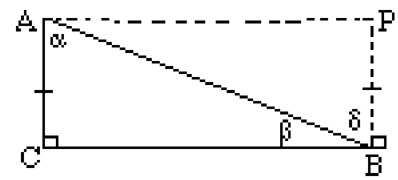


b. The following theorem was proved using the Saccheri Theorem:

Th 1) Given Saccheri quadrilateral ABCD with lower base $\overline{AD}$.

Then m ABD ≤ m CDB.

Using Th 1, complete a sketch proof of Th 2 below:



Th 2) Given right triangle ΔABC with right angle at C and angles as shown in the figure. Then m A + m B ≤ 90. I.e., $\alpha + \beta \le 90$.

Pf) Given ΔABC with right angle at C. Construct a Saccheri quadrilateral as shown. By Th 1, ...

c. The following theorem is proved in <u>Absolute Geometry</u>:

Th 3) The sum of the measures of the angles of any triangle is no more than 180.

Draw a figure, restate the theorem in terms of your figure, and use Th 2 to sketch its proof. (You may use without proof the theorem that the foot of a perpendicular from a vertex to the line containing a longest side of a triangle is between the vertices determining that side. You <u>do</u> have to mention in your proof where you're using this theorem.)

[20 poss: 10, 10]

2. State <u>carefully</u>, and <u>in detail</u>, the Euclidean Parallel Postulate (EPP). Then state carefully, and in detail, the <u>negation</u> of EPP.

[50 possible: 5, 15, 20, 10]

3. Here are three theorems:

(A) Given two lines cut by a transversal. If the lines are parallel, then alternate interior angles are congruent.

(B) Given two lines cut by a transversal. If alternate interior angles are congruent, then the lines are parallel.

(C) Given three lines, l_1, l_2,and l_3. If $l_1 \parallel l_2$, and $l_2 \parallel l_3$, then $l_1 \parallel l_3$.

One of these can be proved in Absolute Geometry and the other two depend on EPP.

a. Which one can be proved in Absolute Geometry?

b. For one of the two theorems that <u>cannot</u> be proved in Absolute Geometry, use the Poincaré model to <u>explain in detail</u> how you know it can't be proved in Absolute Geometry. [A picture alone will not do; you have to present a detailed argument as well. The argument will have to begin "If .... could be proved in ..."]

c. Here is a theorem that was proved in Euclidean Geometry. (We'll call it the Angle Sum Theorem for want of a better name.)

The sum of the measures of the angles of any triangle is 180.

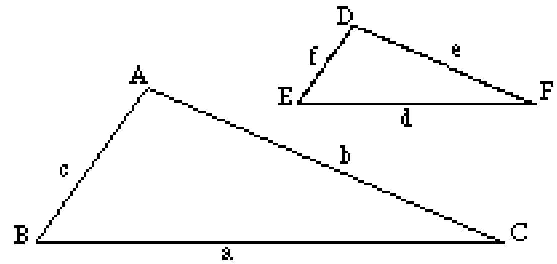
Sketch a proof of the Angle Sum Theorem. [You may use the theorems above in your proof plus any of the other Absolute Geometry theorems we proved.]

d. In Euclidean Geometry, the Exterior Angle Theorem says the an exterior angle of a triangle is the <u>sum</u> of the measures of the two remote interior angles. Draw a figure, restate the theorem in terms of the figure, and sketch its proof.

[40 possible: 10 for a, 30 for b]
4a. On your answer sheet, draw a figure and complete the following definition of similarity for triangles. In your definition, use symbols to express relationships and not words like " ... all pairs of corresponding angles..."]

Def. Given ΔABC and ΔDEF and the correspondence $ABC \leftrightarrow DEF$. Then the correspondence is a similarity, and we write $\Delta ABC \sim \Delta DEF$, iff ...

b. Give a sketch proof of the SAS Similarity Theorem:



Given ΔABC and ΔDEF and the correspondence $ABC \leftrightarrow DEF$.

If A D and $\frac{b}{c} = \frac{e}{f}$, then $\Delta ABC \sim \Delta DEF$.

[Note: If you can't prove the theorem above, for half credit, you may state and prove the Pythagorean Theorem using similar triangles.]

[50 possible: 20 for a, 30 for b]
5a. Give a sketch proof of cases (1) and (2) of the following theorem. [I proved case (1) in class so you are just to reproduce that proof. As a reminder, my proof used the Exterior Angle theorem in Problem 3d.]

Theorem) The measure of an angle inscribed in a circle is half the measure of its intercepted arc.

Restatement) Given circle **C** with center O. If A, B, and C are three points on the circle, then m ABC = $\frac{1}{2} \cdot m(\widehat{AC})$.

There are three cases: (1) O is on one of the sides of ABC; (2) O is the interior of ABC; and (3) O is in the exterior of ABC.

b. Draw a figure corresponding to the following theorem. Then sketch a proof.

Th) Given a circle **C** and two secant lines, l_1 and l_2, which intersect in a point P in the exterior of **C**. Suppose l_1 intersects **C** at the points A and B with P – A – B and l_2 intersects **C** at C and D with P – C – D. Then PA·PB = CP·PD.

[**Note:** For EXTRA CREDIT, you may also prove m BPD = (1/2)·[m($\widehat{BD}$) – m($\widehat{AC}$)] .

For EXTRA CREDIT, answer the questions about circles on pages 3 and 4.

INSTRUCTIONS:

Give your answers on the answer sheets, indicating clearly which problem you're answering. To "sketch a proof" means that you need include only the essential steps and you may use standard notation for parts of triangles (For example, a is the length of the side opposite vertex A, etc.) without stating these things explicitly in your proof. However, when the question is to state an axiom, definition or theorem, you must make everything explicit and not use conventions.

Note: If you're totally at a loss on a proof problem, you can ask me for a hint. I may help you out for some deduction in points for the problem.

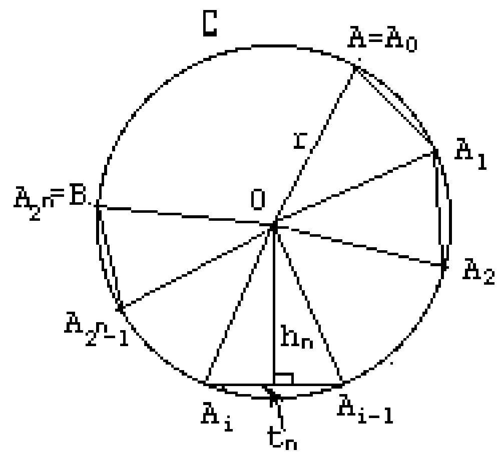
[50 possible]
6. Arc length and area were defined as follows:

Given a circle **C** with center O and radius r. Let $\widehat{AB}$ be an arc of **C**. For each natural number n, take a sequence of $2^n + 1$ points $A_0, A_1, ..., A_{2^n}$ in order on $\widehat{AB}$ with $A_0 = A$, $A_{2^n} = B$, and $A_{i-1}A_i = A_iA_{i+1} = t_n$ for all i.
Also let h_n be the length of the perpendicular from O to $\overleftrightarrow{A_{i-1}A_i}$. Finally, let

$$u_n = \sum_{i=1}^{2^n} A_{i-1}A_i \text{ (which equals } 2^n \cdot t_n.)$$

and

$$\alpha_n = \sum_{i=1}^{2^n} \alpha(\Delta A_{i-1}OA_i) \text{ (which equals } 2^n \cdot \tfrac{1}{2} \cdot t_n \cdot h_n.)$$



Then by definition

$$l(\widehat{AB}) = \lim_{n \to \infty} u_n \quad \text{and} \quad \alpha(\text{sect}AOB) = \lim_{n \to \infty} \alpha_n.$$

a. In the circle on page 4, draw two regular 2^n-pieced broken lines in the major arc $\widehat{AB}$, one for n = 2 and the other for n = 3. Label the vertices of the first broken line A_0, A_1, etc. and the vertices of the second A_0', A_1', etc. Also indicate two segments, one of length h_2 and the other of length t_3, in your figure.

b. Before defining π, we proved the theorem below about the lengths of arcs of two circles. Fill in the missing hypothesis of the theorem.

Th) Given two circles **C** and **C'** with centers O and O' and radii r and r'.

Let $\widehat{AB}$ and $\widehat{A'B'}$ be arcs of the two circles. If _________________, then $l(\widehat{AB}) / 2r = l(\widehat{A'B'}) / 2r'$.

c. To prove the theorem in (c), we considered two regular 2^n-pieced broken lines with vertices $A_0, A_1, ...$ and $A'_0, A'_1, ...$ inscribed in $\widehat{AB}$ and $\widehat{A'B'}$ respectively. A major part of the proof involved showing that $\Delta A_{i-1}OA_i \sim \Delta A'_{i-1}O'A'_i$ for i = 1, 2, ..., 2^n. Draw pictures of the two triangles. Then prove they are similar, telling specifically where you use the hypothesis in the theorem in (b) above.

d. The circumference of a circle is defined to be the length of the major arc $\widehat{AB}$ of the circle when A = B. The theorem in (c) implies that the ratio of the circumference of any circle to twice its radius is a constant. This constant we define to be π. Use this definition to obtain the formula for the circumference of a circle. [Obviously you have to show your one-step derivation.]

e. Use the definition of α(sectAOB) given at the top of page 3 above plus appropriate limit theorems to obtain a formula for α(sectAOB) in terms of $l(\widehat{AB})$. [Again, you must show your work.]

f. Use the formula in (d) to show that the area of a circle (disk) is $\pi \cdot r^2$. [Show your work.]

This is the circle for Problem 7a. Note that you are supposed to be drawing in the <u>major</u> arc $\widehat{AB}$.