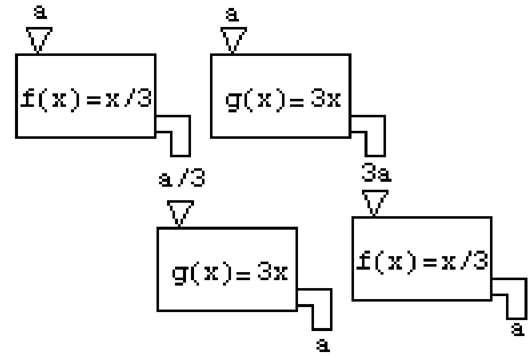
Inverse Functions

Two functions are inverses of one-another iff each undoes what the other does.

Example 1. $f(x) = x/3$ and $g(x) = 3x$.



We have, for example, $f(12) = 4$ and $g(4) = 12$.

Also we have $g(7) = 21$ and $f(21) = 7$. In general,

$f(r) = r/3$ and $g(r/3) = 3\cdot(r/3) = r$; and also

$g(s) = 3s$ and $f(3s) = (3s)/3 = s$.

This means that, if you couple the two functions in either order (as at right), when you perform one after the other, what you put in is what you get out.

Because of this, we say that f and g are inverses of one-another, and we'd write $g = f^{-1}$ (or $f = g^{-1}$.)

Coupling the functions as in the left picture above is done by doing this: $g(f(a)) = g(a/3) = 3\cdot(a/3) = a$.

You can write it for the right picture.

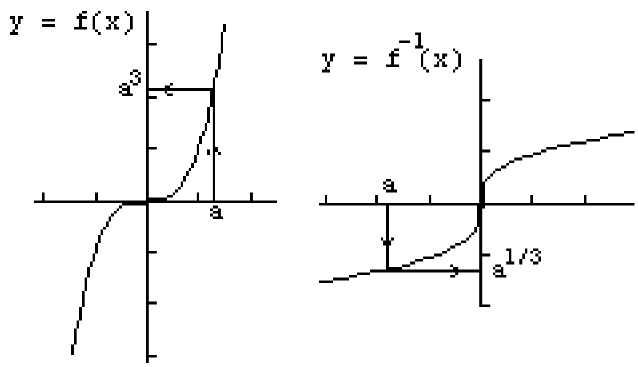
Example 2. $f(x) = x^3$ and $g(x) = x^{1/3}$. Here we have

$f(a) = a^3$ and $g(a^3) = (a^3)^{1/3} = a$;

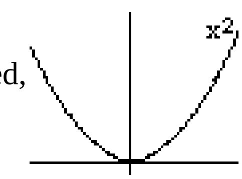
and

$g(a) = a^{1/3}$ and $f(a^{1/3}) = (a^{1/3})^3 = a$.

Hence, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$.

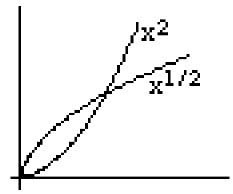


Example 3. However, if $f(x) = x^2$ and $g(x) = x^{1/2}$, then f and g are <u>not</u> inverses of one another. To see this, note that



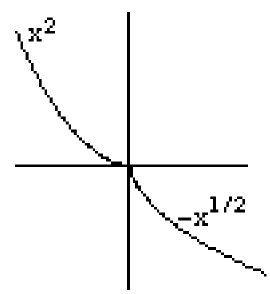
$f(-2) = 4$ but $g(4) = 2$. [Remember, $\sqrt{a}$ is the <u>non-negative</u> number which, when squared, gives a.]

We could make restrictions as follows:



Let $f(x) = x^2$ for $x \geq 0$. Then $g(x) = x^{1/2}$ <u>is</u> the inverse of f.

Or



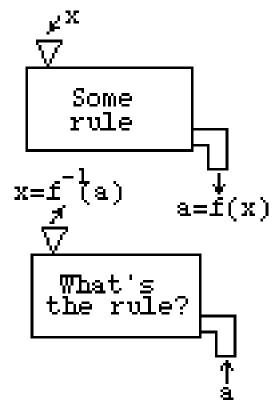
Let $f(x) = x^2$ for $x \leq 0$. Then $g(x) = -x^{1/2}$ is the inverse of f.

In general:

Definition f and g are **inverse functions** iff $f(g(x)) = x$ for every x in the domain of g; and $g(f(x)) = x$ for every x in the domain of f. (Actually, "inverses of one-another" would be better.)

Properties of Inverse Functions

Assume the function f does have an inverse function, f^{-1}.



-- If you have an easy equation for f(x), you can get an equation for $f^{-1}(a)$ by solving for x in the equation f(x) = a.

Reason: The inverse function can be thought of as reversing the original function. If f does something to x to produce a, then f^{-1} will have to do something else to a to produce x.

For example, suppose f(x) = 2x – 1. To find an equation for $f^{-1}(a)$, solve 2x – 1 = a to get x = (a + 1)/2. Thus

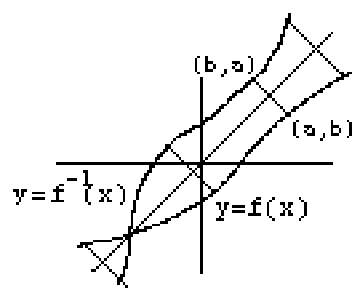
$f^{-1}(a) = (a + 1)/2$. (You could also write this as $f^{-1}(x) = (x + 1)/2$ since the choice of variable is irrelevant.)

-- $f^{-1}(r) = s$ iff f(s) = r.

For example, since $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$ are inverse functions, then $r^{1/3} = s$ iff $s^3 = r$.

-- If (a, b) is on the graph of y = f(x), then (b, a) is on the graph of $y = f^{-1}(x)$; and, conversely, if (a, b) is on the graph of $y = f^{-1}(x)$, then (b, a) is on the graph of y = f(x).

For example, with the functions f(x) = 2x – 1 and $f^{-1}(x) = (x + 1)/2$, if (a, b) is on the graph of y = 2x – 1, then (a, b) satisfies the equation so b = 2a – 1. But then a = (b + 1)/2 so (b,a) satisfies the equation y = (x + 1)/2. Hence (b, a) is on the graph of y = (x + 1)/2.



In general: If (a, b) is on the graph of y = f(x), then (a, b) must satisfy the equation so b = f(a). But then $a = f^{-1}(b)$ so (b, a) satisfies the equation $y = f^{-1}(x)$. Therefore, (b, a) is on the graph of $y = f^{-1}(x)$.

The implication is that the graphs of y = f(x) and $y = f^{-1}(x)$ are symmetric about the 45° angle line, y = x. *

-- Other examples of functions that are inverses of one-another:

$y = e^x$ and y = ln x (And, in general, $y = b^x$ and $y = \log_b x$.)

y = sin x (domain $\{x \mid -\pi/2 \le x \le \pi/2\}$) and $y = \sin^{-1}x$.

y = cos x (domain $\{x \mid 0 \le x \le \pi\}$) and $y = \cos^{-1}x$.

y = tan x (domain $\{x \mid -\pi/2 < x < \pi/2\}$) and $y = \tan^{-1}x$.

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* The reason is that the points (a, b) and (b, a) are mirror images of one-another about the line y = x. The reason for **that** is (1) the line containing the points (a, b) and (b, a) is perpendicular to the line y = x (compute the slopes to see this); and (2) the midpoint of the line containing the points (a, b) and (b, a) is on the line y = x. (Use the midpoint formula to calculate the midpoint and then see that the coordinates satisfy the equation y = x.)