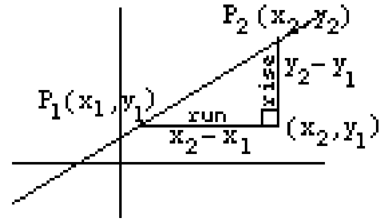
LINES

Slope The slope of a nonvertical line in a coordinate plane is defined as follows:

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on the line. Then

$$\text{slope of the line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in y}}{\text{change in x}} = \frac{\text{rise}}{\text{run}}.$$

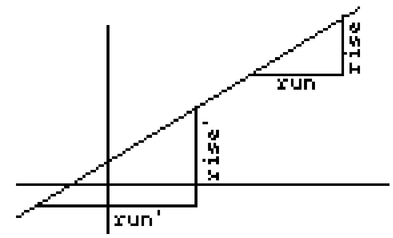
A vertical line has no slope.



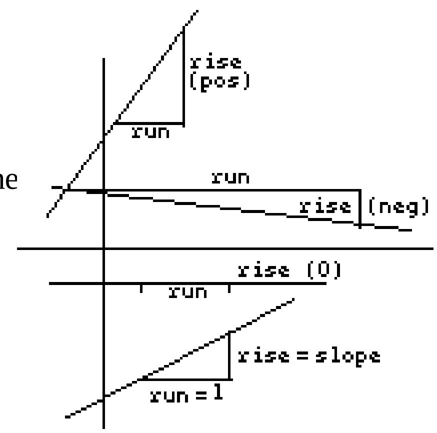
Fact: A nonvertical line has only one slope. That is, it makes no difference which pair of points you choose on the line to compute the slope because you'll always get the same result.

Reason: The triangles at right are similar so

$$\frac{\text{rise}}{\text{run}} = \frac{\text{rise}'}{\text{run}'}.$$

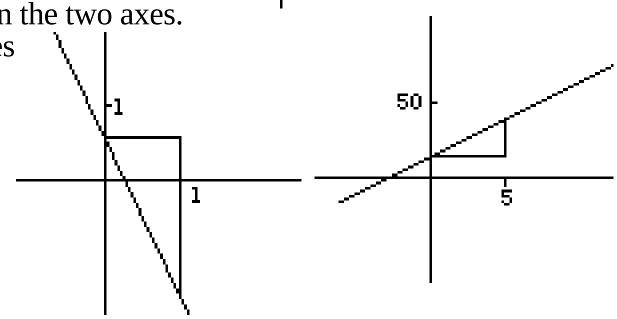


When finding (or estimating) slopes, if you take the run to be positive, then the "rise" can be positive, negative, or zero. Also, if you take the run to be 1, then the rise is actually the slope of the line.

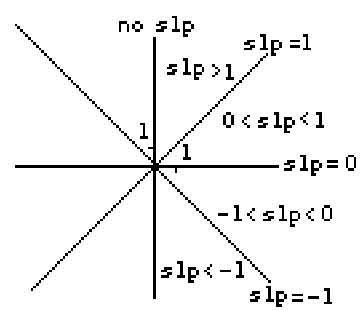


When estimating slopes, you have to pay attention to the scales on the two axes. If the scales are the same, slopes are easy to estimate. If the scales are different, you'll have to use the scales to estimate the run and the corresponding rise.

For example, in the first picture at right, the scales are the same. For a run of 1, the "rise" is about –2 so the slope is about –2. In the second picture, the scales are not the same. With the line shown, for a run of 5, the rise is about 20 (by eyeball) so the slope is about 4.

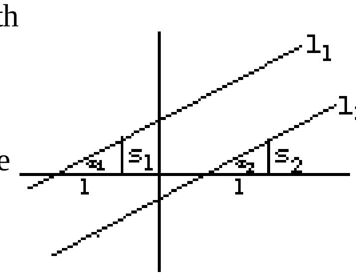


Since most coordinate systems have equal scales, you should be able to approximate slopes of lines by eyeball. The basic slopes for lines thru the origin are shown at right. The slope for a line not thru the origin is easy to estimate by visualizing a parallel line which does contain the origin.



Fact: Nonvertical parallel lines have the same slope. Conversely, nonvertical lines with the same slope are parallel.

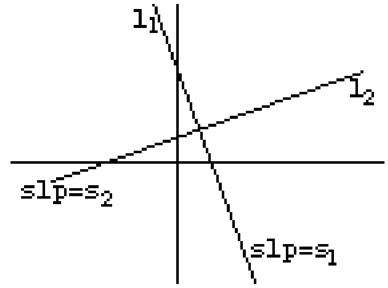
Reason: l_1 and l_2 nonvertical lines with slopes s_1 and s_2. If $l_1 \parallel l_2$, then $a_1 = a_2$ so the triangles are congruent and therefore $s_1 = s_2$. If, conversely, $s_1 = s_2$, then the triangles are congruent so $a_1 = a_2$ and therefore $l_1 \parallel l_2$. [This argument assumed the slopes were positive but the same conclusion holds when they're zero or negative.]



Fact: Two nonvertical lines are perpendicular if, and only if, their slopes are negative reciprocals of one another.

In the figure at right, l_1 l_2 iff $s_1 \cdot s_2 = -1$.

Reason: This is harder. I'll give an argument in an Addendum to this handout.



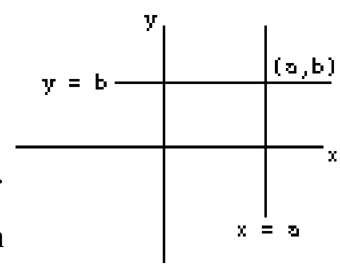
After all this fuss about slopes, you probably wonder why slopes are so important. We'll return to that question in a bit -- after looking at equations for lines.

Equations for Lines We'll take the following as axiomatic:

Basic Fact: Any equation that can be put in the form Ax + By = C (A,B,C are constants, and not both A and B are zero) has a graph which is a line. Conversely, every line is the graph of an equation that can be put in this form. [This is often called the general form for the equation of a line.]

Fact: Given the point (a, b). The equation x = a is the equation of the vertical line thru the point (a, b), and the equation y = b is the equation for the horizontal line thru the point (a, b).

Reason: Written in the general form, these equations are 1x + 0y = a and 0x + 1y = b. Both equations therefore have graphs that are lines. If you think about the ordered pairs that make the equation 1x + 0y = a true, you should be able to see why the graph is the vertical line thru (a, b). Similarly for the equation y = b.

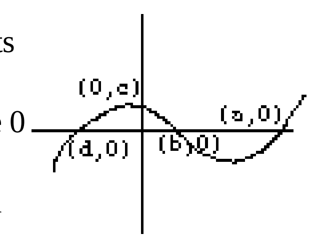


Some general things about graphs:

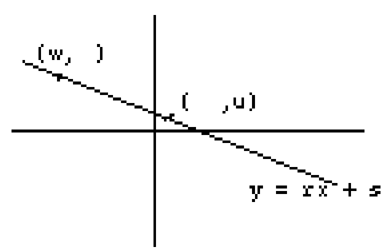
(1) A y-intercept is a point where the graph intersects the y-axis. To find the y-intercepts from an equation, put 0 in for x and solve for y.

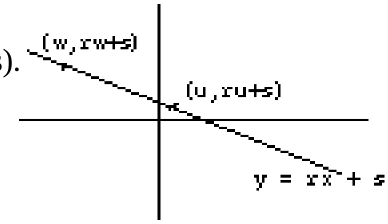
An x-intercept is a point where the graph hits the x-axis. To find x-intercepts, substitute 0 for y and solve for x.

[To remember which variable gets replaced by zero, remember what intercept means and which variable must be zero as in the figure.]



(2) The graph of an equation is the set of all ordered pairs that satisfy the equation. For example, given the equation y = rx + s. The point with x-coordinate w on the graph of this equation has y-coordinate rw + s. That is, the point (w, rw + s) is on this line. Similarly, the point with y-coordinate u which is on the graph has x-coordinate (u – s)/r. That is, the point ((u – s)/r , u) is on the line.





Fact: An equation that can be put in the form $y = rx + s$ (r, s are constants) has a graph that is a line. The slope of the line is the number r and the y-intercept is (0, s).

Reason: The equation can be written in the form $rx + (-1)y = -s$ so its graph is a line by the Basic Fact on page 2. Next take two points on the line, say $(w , rw + s)$ and $(u , ru + s)$. If you use these two points to compute the slope [Don't forget parentheses!], the result will be the number r. [Do it.] And to find the y-intercept, substituting 0 for x in the equation gives s for y.

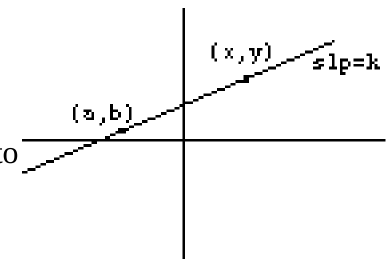
The form $y = rx + s$ is called the slope-intercept form for the equation of a line. **When you are asked to find the equation of a line, your result should be given in the slope-intercept form unless otherwise specified**.

[Note: There is nothing special about the constants r and s -- nor about m and b. (Some of you might think that "m" is synonymous with "slope". It is not.) The important thing is that, when an equation is in the slope-intercept form, the coefficient of x is the slope and the constant term is the y-value of the y-intercept.]

Things you must be able to do:

1. Find the equation of the line thru a given point with a given slope.

Example: Find the equation of the line thru the point (a, b) with slope k.



Here's how:

(1) Draw a rough picture. [Important!]

(2) Take an "arbitrary" (i.e., general) point (x, y) on the line. [This is supposed to represent any point on the line.]

(3) Use the arbitrary point and the given point to write the slope of the line:

(4) Set this equal to the given slope. (A line has only one slope so the slope $\frac{y-b}{x-a}$ you calculated must be the given slope k.)

$$\frac{y-b}{x-a} = k$$

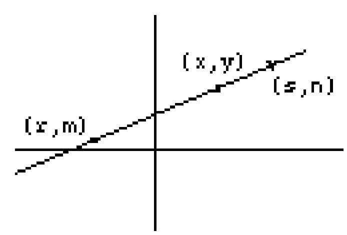
(5) Simplify the equation, writing the result in the slope-intercept form:

$$y - b = k(x - a) \quad \text{iff} \quad y = kx - ka + b \quad \text{iff} \quad y = kx + (b - ka).$$

(6) **CHECK.** [This is very important. And it does not mean to look in the answer book to see if you agree with the book!] To check in the example, the slope is the coefficient of x which is k, and that is what was given. Also, when $x = a$, $y = ka + (b - ka) = b$ so the point (a, b) is on the line as was given.

2. Find the equation of the line thru two given points.

Example. Find the equation of the line thru the points (r, m) and (s, n).



[If r = s, the equation is x = r. Do you see why?]

(1) Draw a rough picture.

(2) Use the two given points to compute the slope of the line:

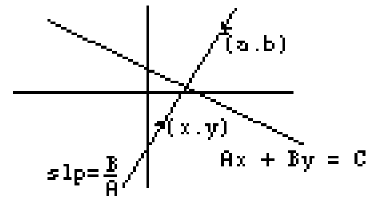
$$\text{slope} = \frac{n-m}{s-r}$$

(3) Now use this slope and either of the two given points to find the equation as in problem **1**.

$$\frac{n-m}{s-r} = \frac{y-r}{x-m} \quad \text{iff} \quad y = \frac{n-m}{s-r}(x-m) + r \quad \text{iff etc.}$$

(4) Check. [Do the two given points satisfy the final equation?]

3. Variations on the basic theme.



Example 1: Find the equation of the line thru the point (a,b) which is perpendicular to the line with equation Ax + By = C ($B \neq 0$).

(1) Draw a picture.

(2) Find the slope of the given line by putting the equation in the slope-intercept form: $y = -\frac{A}{B}x + \frac{C}{B}$

(3) The slope of the desired line is therefore B/A. Now use this slope and the given point to find the equation of the desired line.

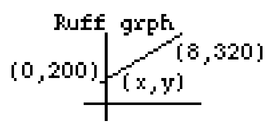
Example 2: Car sales at Acme Used Car Co. have increased linearly since the base year of 1980 when 200 cars were sold. In 1988, a total of 320 cars were sold. (a) Write an equation that gives the number of cars sold each year since 1980. (b) Draw a graph expressing this relationship. (c) How many cars will be sold in 1992, assuming a continued linear growth? (d) In what year will 470 cars be sold, assuming a continued linear growth?

Solution: The key here is that the problem says there's been a linear increase which means that we should find a linear equation that relates the two quantities in question. As always, define the variables.

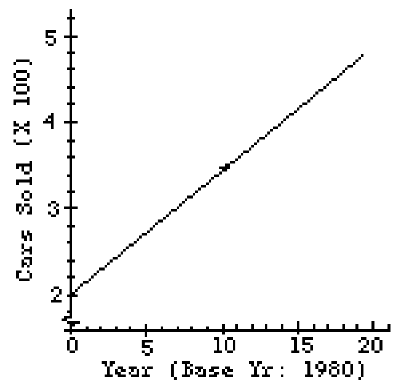
Let y be the number of cars sold x years after the base year of 1980. We know that when x = 0, y = 200 and when x = 8, y = 320. We want the equation of the line thru (0,200) and (8,320).

Slope is $\frac{320 - 200}{8 - 0} = \frac{120}{8} = 15$

$\frac{y - 200}{x - 0} = 15 \qquad y - 200 = 15x$



$y = 15x + 200$ is the equation.



We have the equation so now we'll draw the graph.

[Comments: Label axes on your graphs when doing an applied problem. Use all of the coordinate systems that makes sense for the problem when choosing your scale. Do not include unnecessary numbers - like zero to 200 on the y-axis. To indicate you're omitting numbers, use  as I did.

(c) The question is: When x = 1992 – 1980 = 12, what's y?

Answer: $y = 15 \cdot 12 + 200 = 180 + 200 = 380$. About 380 cars will be sold in 1992, assuming continued linear growth.

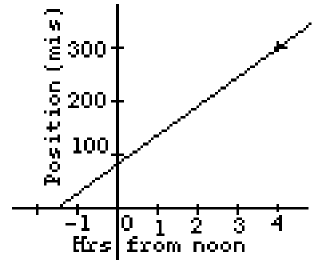
(d) The question is: When y = 470, what's x?

Answer: Solve $470 = 15x + 200$ to get $x = 18$. The year is 18 + 1980 = 1998.

470 cars will be sold in 1998, again assuming continued linear growth.

Example 3. I'm driving on the interstate at a constant speed of 55 miles per hour. At noon, I see that I've just passed mile marker number 82.5 (and the next marker is 82.6) Write an equation that tells my position on the highway using noon as the base time.

Solution: Let s be my position t hours from noon. Since speed is constant, then distance = rate·time so my position will be given by the equation $s = 55t + 82.5$. A graph of the equation is given at right.



The Significance of Slope

A linear equation gives a relationship between two quantities, the independent quantity (x) and the dependent quantity (y). The slope of the line tells the rate of change of the dependent quantity with respect to the independent quantity. It does this by telling the change in y for a unit change in x. Some examples:

Look at Example 3 above. If you use the points (0, 82.5) and (2, 192.5) to compute the slope, you're really finding the average speed from time t = 0 to time t = 2 because you're computing

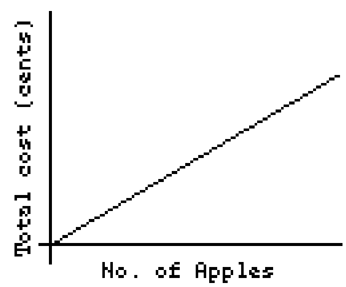
(192.5 – 82.5)/(2 – 0) = distance / time = average speed.

Similarly, if you use the points say (1, 137.5) and (4, 302.5), you'll get the average speed between 1 p.m. and 4 p.m. No matter which pair of points you use, you'll get 55 which is the average speed for any interval of time on this trip. (The reason is that the speed was constant.) Please note that, since the position was measured in miles and the time in hours, then the rate of change is given in miles per hour.

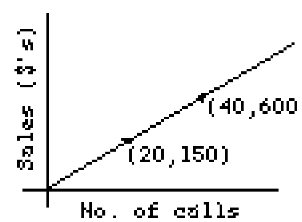
Look at Example 2 on the previous page. Using any pair of points on the graph to compute the slope, the value will always be 15. What this means is that, for any year after 1980, Acme's sales have increased at the rate of 15 cars per year. Note again that the dependent variable is number of cars sold and the independent variable is years so the slope, the rate of change, is measured in number of cars sold per year.

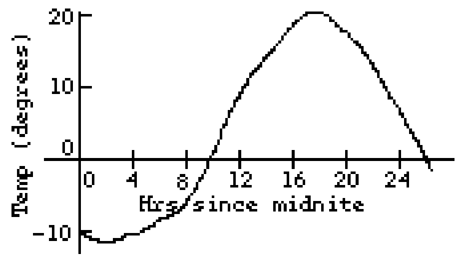
Here are a few more examples.

Apples are 25 cents each. y is the cost of x apples. The equation relating x and y is $y = 25x$. The slope is 25. It means that the rate for apples is 25 cents per apple.



A telemarketer finds that her sales increase linearly as a function of the number of calls she makes. The graph is given at right. Using the two given points to compute the slope, we get slope = 22.5. This means that she averages $22.50 in sales per call.





Yesterday, the time and temperature were given by the graph at right. Suppose you're interested in the average rate if change of the temperature between 4 a.m. and 10 a.m. Using the approximate points (4, –10) and (10, 2), we get

$$\text{Avgtemp change} = \frac{2-(-10)}{10-4} = 2$$

This means that, between 4 a.m. and 10 a.m., the temperature rose an average of 2 degrees per hour. If we're concerned about how the temperature changed between 4 p.m. and midnight, we'd use the (estimated) points (16, 19) and (24, 7) to compute

$$\text{Avg tempchange} = \frac{7-19}{24-16} = -1.5$$

Thus between 4 p.m. and midnight, the temperature dropped (note the negative) at the average rate of 1.5 degrees per hour.

In calculus, you will learn to find <u>instantaneous</u> rate of change of y with respect to x. This means, for example, that you will find the rate at which the temperature is changing at exactly 1 p.m. Or for a car that is <u>not</u> traveling at a constant speed, you will learn to compute the car's speed at a given <u>instant</u> in time. This is much more realistic since it is the instantaneous speed that your speedometer gives, not an average speed. To do these things, you will be computing <u>the slope of the tangent line to the curve</u> at a particular point on the curve. And <u>that</u> is why slope is so important!

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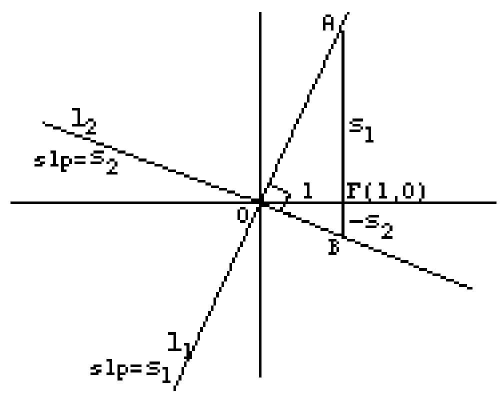
Addendum



The following is an argument for this fact:

Given lines l_1 and l_2 with slopes s_1 and s_2. Then l_1 l_2 iff* $s_1 \cdot s_2 = -1$.

I'll show only that if l_1 l_2, then $s_1 \cdot s_2 = -1$. Also, I'll only argue for the case when the lines intersect at the origin. Since parallel lines have the same slope, it's not hard to extend this argument to the case where the lines intersect elsewhere.

Suppose that l_1 and l_2 are perpendicular and intersect at the origin. Assume also that l_1 has positive slope s_1. Then the slope, s_2, of the other line "obviously"** must be negative. In the figure at right, $\overline{AB}$ is the vertical line thru the point (1, 0). Then AF = s_1 and FB = $-s_2$.*** Since m AOF = 90 – m A = m B, then $\Delta AOF \sim \Delta OBF$ so

$$\frac{AF}{OF} = \frac{OF}{BF} \quad \text{or} \quad \frac{s_1}{1} = \frac{1}{-s_2}.$$ Hence $s_1 \cdot s_2 = -1$.

Footnotes:

* The symbol "iff" is a shorthand way of writing "if, and only if,". Thus l_1 l_2 iff $s_1 \cdot s_2 = -1$ means l_1 l_2 if, and only if, $s_1 \cdot s_2 = -1$. This in turn means "If l_1 l_2, then $s_1 \cdot s_2 = -1$ and if $s_1 \cdot s_2 = -1$, then l_1 l_2."

** Always be suspicious when someone says "obviously."

*** The number FB is the length of the side of a triangle so must be positive. Since s_2 is a negative number, its opposite, which is denoted by $-s_2$, must be a positive number. [You should read "–x" as "the opposite of x" rather than "negative x" since –x could be a positive number. It really seems strange to say "negative x is a positive number."]