MA-161 (F,07)
Final Exam

Name___________________________

The instructions are on a separate page. Read them first.

[40 possible: 20, 20]

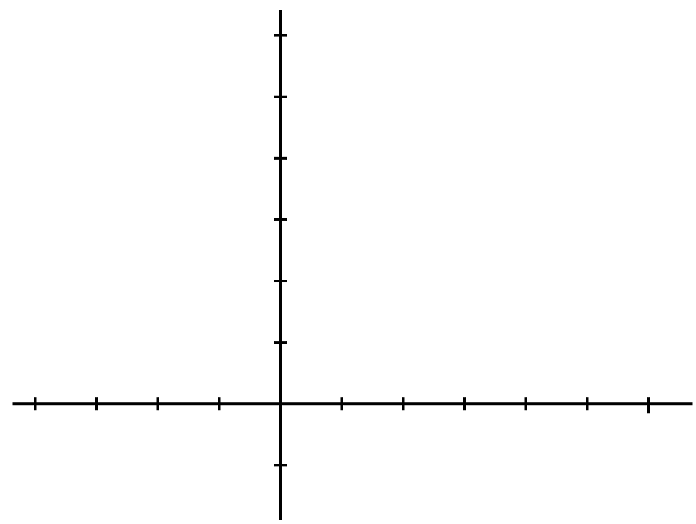
1. [For this problem, both (a) and (b), **do all your work and give your answers on this page and the next below the line.**] (Problem 1,b is on the next page.)

For the function $f(x) = \frac{1}{4}(x-2)^2 + 1$, do the following:

a. i. Tell **exactly** (not approximately) the slope of the tangent line to the graph at the point where x = 1.

ii. Give the equation of the tangent line.

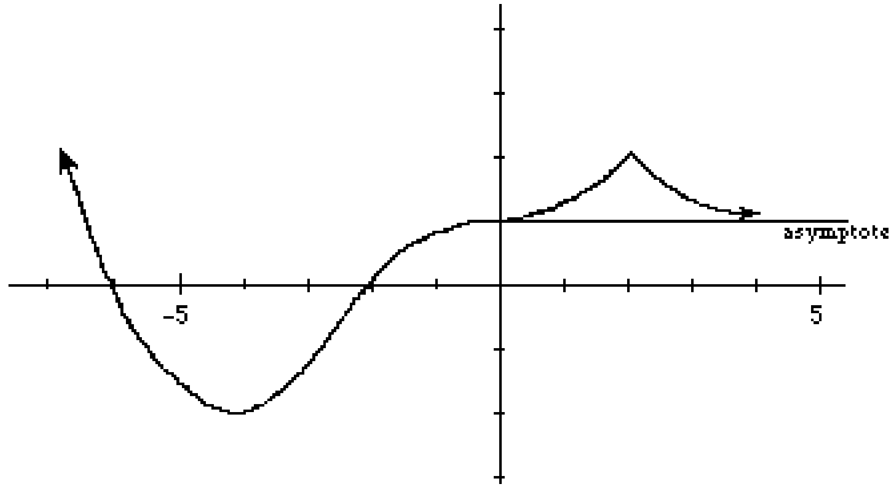
iii. Draw a careful graph of the function and the tangent line on the coordinate system below.



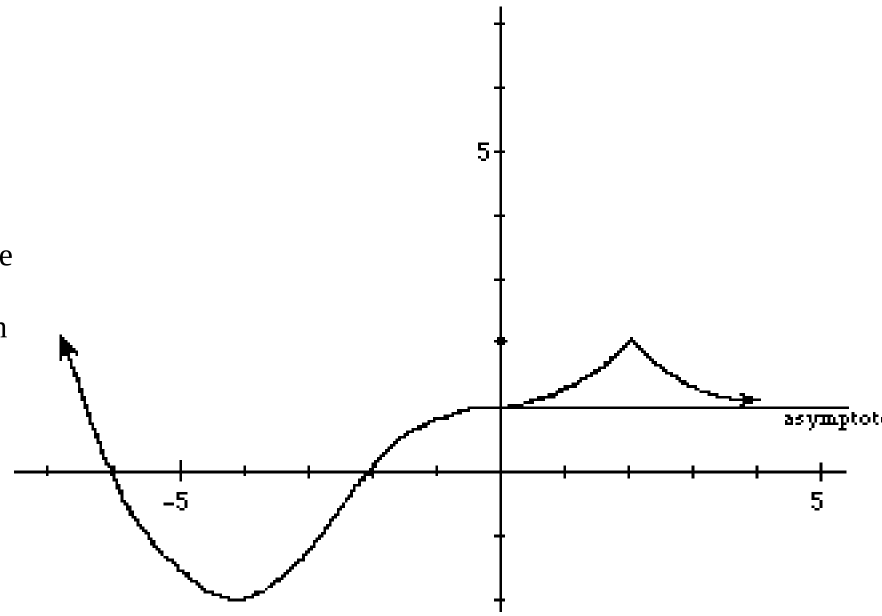
1. (Continued) Again, **do all your work and give your answers on this page below the line.**]

b. Tell the **exact** (not approximate) area of the region bounded on the top by the graph of $f(x) = \frac{1}{4}(x-2)^2 + 1$, on the bottom by the the x-axis, on the left by the line $x = -1$, and on the right by $x = 4$. [**Note:** You must include an appropriate picture. Also, you might want to check your result by some other means to see that you haven't made an error.]

[10 possible]
2a. In the figure at right is the graph of $y = f(x)$. Use it to draw a careful graph of the graph of the **derivative**, $y = f'(x)$. Draw your graph for values of x in the interval $[-7, 5]$. Be sure your picture is consistent with the important features of the original function.
[Draw your picture at right.]



[10 possible]
b. In the figure at right is the graph of $y = f(x)$. Use it to draw a careful graph of the **antiderivative**, $y = F(x)$, of f **that goes through the point (0, 2)**. Draw your graph for values of x in $[-7, 5]$. Be sure your picture is consistent with the important features of the original function.
[Draw your picture at right.]



The rest of the problems are to be done on the blank paper I've given you and not on the test paper.

[15 possible]
3. Here are three basic limits that were used in obtaining derivative formulas:

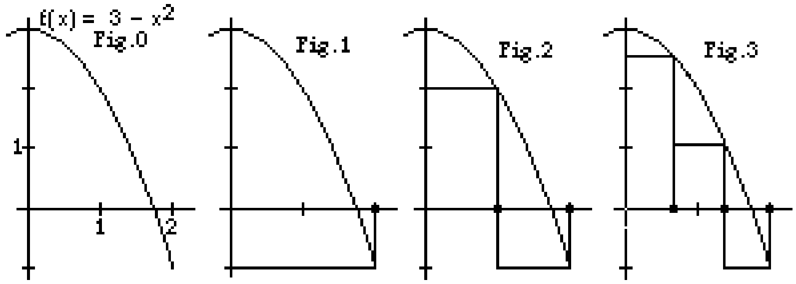
(a) $\lim\limits_{w \to 0} \dfrac{e^{w} - 1}{w}$ (b) $\lim\limits_{w \to 0} \dfrac{\sin w}{w}$ (c) $\lim\limits_{w \to 0} \dfrac{\cos w - 1}{w}$

For <u>one</u> of these limits, estimate its value. <u>Show in detail how you obtained your estimate</u>. Your explanation will require words plus possibly a table or a graph or ...

[20 possible]
4. Use the **definition of derivative** and the limit facts to derive the formula for <u>one</u> of these derivatives:

(a) $\dfrac{d}{dx}\left(e^{x}\right)$ or (b) $\dfrac{d}{dx}\left(\sin x\right)$ or (c) $\dfrac{d}{dx}\left(x^{n}\right)$ (n is a natural number.)

If you can't do any of these, you may, for 85% credit, derive the formula for $\dfrac{d}{dx}\left(k \cdot x^{2}\right)$ (k is a constant)



[20 possible: 5, 15]

5. Consider the function $f(x) = 3 - x^2$ on the interval [0, 2] (Fig. 0). Suppose we compute a sequence of numbers A_1, A_2, A_3, ... as follows:

(Fig. 1) $A_1 = 2\cdot f(2) = 2\cdot(-1) = -2$;

(Fig. 2) $A_2 = 1\cdot f(1) + 1\cdot f(2) = 1\cdot 2 + 1\cdot(-1) = 1$

(Fig. 3) $A_3 = \frac{2}{3}\cdot f\left(1\cdot\frac{2}{3}\right) + \frac{2}{3}\cdot f\left(2\cdot\frac{2}{3}\right) + \frac{2}{3}\cdot f\left(3\cdot\frac{2}{3}\right) = \frac{2}{3}\cdot\frac{23}{9} + \frac{2}{3}\cdot\frac{11}{9} + \frac{2}{3}\cdot(-2) = \frac{50}{27} = 1\,\frac{23}{27}$; and so forth.

This sequence of numbers A_1, A_2, A_3, ... approaches a limit. The Fundamental Theorem of Calculus (FTC) says that this limit can be found by ...

Here's your problem:

a. Draw a picture and give the value of the number A_4, the fourth term in the sequence above. Show your work as I've done above.

b. Describe carefully **how the FTC says the limit of the sequence can be obtained**. Then do what you've just described and actually compute the exact value of $\lim_{n\to\infty} A_n$. (If you know what the FTC says, this part of problem should take no more than a few minutes. And be sure you have answered the question!)

[20 possible: 6, 6, 4, 4]

6. In the table below, f(t) is the number of milligrams of a product that has been produced in a chemical reaction t minutes after the reaction begins. (Answer the questions in sentences, including units of measurement. And please remember that you have to **show your work** and **explain** what you did to get your result. A rough labeled graph might help you to understand the problem -- but the graph should <u>not</u> be used to get the values requested below.)

t (in minutes)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
f(t) (amount produced in mgs)	0.25	1.33	2.16	2.70	3.11	3.40	3.61	3.75	3.86

(a) What is the **average** rate at which the product is being produced between 0.25 and 1.00 minutes after the reaction begins? (b) Approximate the **instantaneous** rate of change in production at exactly t = 1.00 minute after the reaction begins. (c) Is the instantaneous rate of change increasing or decreasing over time? What does this imply about the concavity of the graph of f?

[20 possible: 5 each]

7. Find these derivatives, anti-derivatives and definite integrals. You'll have to rewrite the original problem on your answer sheet. (a, b, and n are all constants.) [Remember the Chain Rule!]

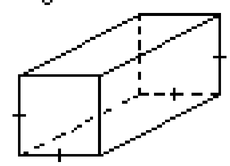
a. $\frac{d}{dx}\left(\frac{e^{ax}}{b}\right)$ and $\frac{d^2}{dx^2}\left(\frac{e^{ax}}{b}\right)$ b. $\int \left(\sin^n(ax+b)\right)\cdot\cos(ax+b)\,dx$ c. $\int_0^1 x\cdot e^{-\frac{1}{2}x^2}dx$

[40 possible points]
8. [Use methods of calculus to do this problem, showing all your work. Also, **you might reread the instructions concerning extreme value problems**. Use the Second Derivative Test in this problem so that I can tell if you know what it says and how to use it. If you can't do the given problem, you may do Problem (A) for 90% credit, (B) for 85% credit, or (C) for 75% credit.]

An architect is confronted with the problem of constructing a rectangular room with K square feet of floor space. (K is a positive constant.) If she wants to construct the room so that the floor has the smallest possible perimeter, what should the width and length of the room be? What is the perimeter in that case?

Alternative problems for less credit: [Note: Even if you did the problem above, you might want to do one of the following problems in case you made a major error on the one you did.]

Fig for Alt Prob A



A. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length. The frame is to be made from a wire of length L feet. (L is a positive constant.) What should the dimensions of the frame be to obtain a box of greatest volume, and what is that maximal volume?

B. Think of the graph of the equation $y = \sqrt{x}$. Find the point on the graph which is closest to the point (3, 0).

C. An open box is to be made from a 15 inch by 24 inch rectangular sheet of metal by cutting identical squares from each corner and folding up the sides. What are the dimensions and the volume of the largest possible box (that is, the box with largest volume) that can be made from this sheet of metal?

[40 possible: 8, 7, 15, 10]
9. [You will need more than one picture for this problem since there is no partial credit without the appropriate pictures. Also, include units of measurement in your answers. Finally, it should go without saying that you should immediately check any antiderivatives you use in this, and any other, problem.]

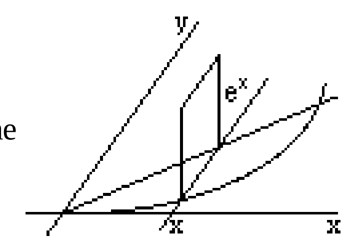
On a single coordinate system, draw graphs of $y = x^2$ and $y = 2x$. The graphs intersect at the points (0, 0) and (2, 4). Let R be the region between the two curves. Do the following:

a. Find the area of R.

b. Find the volume of the solid obtained when R is revolved about the x-axis.

c. Set up the integrals for the volume of the solid obtained when R is revolved about (i) the y-axis; (ii) the vertical line $x = -2$; and (iii) the horizontal line $y = 4$. Do **not** evaluate these integrals.

d. R is the base of a solid. Every cross section of the solid by a plane perpendicular to the x-axis is a rectangle with height $h(x) = e^x$. (i) Set up a definite integral that gives the volume of this solid. (ii) You probably cannot find an antiderivative to evaluate the definite integral. However, you can give can give a numerical value for the volume, correct to two decimal places. Do so. And be sure you have answered the question.



[20 possible. Part a is worth 2 pts; b is 2 pts; c is 6 pts; **d is 10 pts**.]

10. [The graph for this problem is below. Do all the drawing, shading, etc. on this graph. **Do your work, and answer the questions, on the blank paper I've given you and not on this test paper.** Remember to <u>answer in sentences, including units of measurement.</u>]

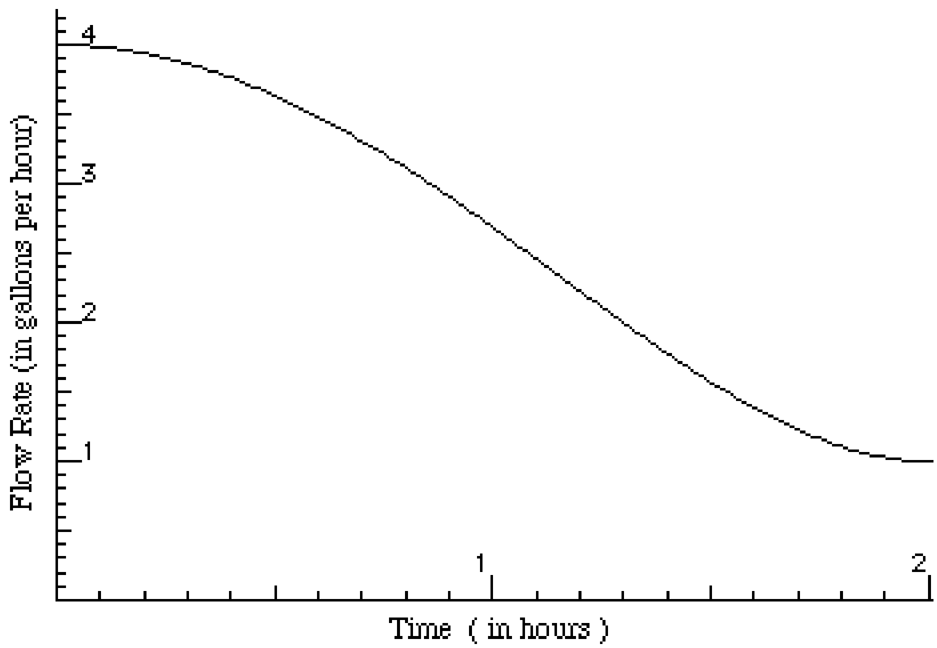
Water is flowing into a tank at the rate of R(t) gallons per hour where t is measured in hours. The graph of R(t) is given below.

a. Write a definite integral that expresses the total amount of water has flowed into the tank in the first hour and fifteen minutes.

b. On the graph, shade the region whose area represents the total amount of water that has flowed into the tank in the first hour and fifteen minutes.

c. Give a numerical approximation for the total amount of water that has flowed into the tank in the first <u>two hours</u>. To do so, divide the interval into **half hour** subintervals and use **midpoints**.

d. One of the numbers that you will have computed in answering (c) is $(.5)\cdot R(1.25)$. Considering the meaning of the .5 and R(1.25) in the context of this problem about the rate of water flow over time, explain in detail what the number $(.5)\cdot R(1.25)$ has to do with approximating <u>the **total amount** of water that has flowed into the tank</u>? In answering this question, you must assume that I have no idea why the area under this graph tells something about the volume of water that has flowed into the tank. In fact, the word "area" should not appear in your explanation.



Problem 11 is to be done on the blank paper I've given you and not on the test paper.

[20 possible]

11. By methods you'll learn later, the graph in Problem 10 can be approximated by the fourth degree polynomial equation at right:

$$R(t) = \frac{3}{16}t^4 - \frac{3}{2}t^2 + 4 \quad \text{for} \quad 0 \le t \le 2.$$

a. Knowing this, you can use your calculator to give an approximation for the total amount of water that has flowed into the tank in the first two hours. Do so, using 50 subdivisions. [Reread the general intructions about using your program. Be sure to answer the question.]

b. Also, knowing a formula for the graph, you can use the Fundamental Theorem of Calculus to find the total amount of water that has flowed into the tank in the first two hours. Use the FTC to do so, showing your work.

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If you have the time and the stamina, you can ask me for the sheet of EXTRA CREDIT problems. But do so only after you've completed everything you can do on Problems 1 - 11 and you've checked over your work.