Math 163 - Calculus - Exam 3 - Practice Solutions
November 1, 2026

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1. Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^{+\infty} \frac{(2x)^n}{n}$.

   Solution: Ratio test.

   $$\left|\frac{(2x)^{n+1}/(n+1)}{(2x)^n/n}\right| = \frac{n}{n+1}\,2|x| \longrightarrow 2|x|.$$

   The series converges when $2|x| < 1$, i.e. $|x| < \frac{1}{2}$. So $\mathbf{R = \frac{1}{2}}$.

   Endpoints. The ratio test gives 1 there, so plug them in:

   - $x = \frac{1}{2}$: $\displaystyle\sum \frac{1^n}{n} = \sum \frac{1}{n}$, the harmonic series. *Diverges.*
   - $x = -\frac{1}{2}$: $\displaystyle\sum \frac{(-1)^n}{n}$. By the AST ($\frac{1}{n}$ decreases to 0), it *converges.*

   Interval of convergence: $\left[-\frac{1}{2},\ \frac{1}{2}\right)$.

2. Find the radius of convergence of $\displaystyle\sum_{k=1}^{+\infty} \frac{(k!)^2}{(2k)!}\,x^k$.

   Solution: Use $(2k+2)! = (2k+2)(2k+1)\,(2k)!$ and $\big((k+1)!\big)^2 = (k+1)^2(k!)^2$:

   $$\left|\frac{c_{k+1}}{c_k}\right| = \frac{\big((k+1)!\big)^2}{(2k+2)!}\cdot\frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{k+1}{2(2k+1)} \longrightarrow \frac{1}{4}.$$

   So the series converges when $\frac{1}{4}|x| < 1$, giving $\mathbf{R = 4}$.

3. Use $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$ to find a power series for $f(x) = \dfrac{x^2}{1-3x^2}$, centered at $a = 0$, and find its interval of convergence.

   Solution: Substitute $u = 3x^2$, then multiply by x^2:

   $$\frac{x^2}{1-3x^2} = x^2\sum_{n=0}^{\infty}(3x^2)^n = \sum_{n=0}^{\infty} 3^n x^{2n+2}.$$

   This converges when $|3x^2| < 1$, i.e. $|x| < \frac{1}{\sqrt{3}}$. At $x = \pm\frac{1}{\sqrt{3}}$ the terms are $3^n \cdot \frac{1}{3^{n+1}} = \frac{1}{3}$, which don't go to 0, so both endpoints diverge.

   Interval of convergence: $\left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$.

4. Find a power series for $f(x) = \dfrac{1}{6-x}$ centered at $a = 5$, and find its interval of convergence.

Solution: Rewrite the denominator in terms of $(x-5)$: $6-x = 1-(x-5)$. So

$$\frac{1}{6-x} = \frac{1}{1-(x-5)} = \sum_{n=0}^{\infty}(x-5)^n.$$

This converges when $|x-5|<1$. At both endpoints the terms are ± 1, which don't go to 0.
Interval of convergence: $(4,\ 6)$**.**

5. Use partial fraction decomposition to find a power series for $f(x) = \dfrac{2}{(x-3)(x+1)}$ centered at $a=0$.

Solution: Partial fractions. $2 = A(x+1) + B(x-3)$. Setting $x=3$ gives $A=\frac{1}{2}$, and $x=-1$ gives $B=-\frac{1}{2}$:

$$f(x) = \frac{1/2}{x-3} - \frac{1/2}{x+1}.$$

Expand each piece (get a 1 in front of each denominator):

$$\frac{1}{x-3} = -\frac{1}{3}\cdot\frac{1}{1-\frac{x}{3}} = -\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}, \qquad \frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Combine:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{2}\left[(-1)^{n+1} - \left(\frac{1}{3}\right)^{n+1}\right]x^n, \qquad |x|<1.$$

Check: $f(0) = \frac{2}{(-3)(1)} = -\frac{2}{3}$, and the $n=0$ coefficient is $\frac{1}{2}\left(-1-\frac{1}{3}\right) = -\frac{2}{3}$.

6. Differentiate $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{+\infty} x^n$ term by term, and use the result to find the exact value of $\displaystyle\sum_{n=1}^{+\infty}\frac{n}{3^n}$.

Solution: Differentiating term by term:

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^{n-1}.$$

The sum we want has exponent n, not $n-1$, so multiply by x:

$$\sum_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2}.$$

At $x=\frac{1}{3}$ (inside the radius of convergence, $R=1$):

$$\sum_{n=1}^{\infty}\frac{n}{3^n} = \frac{1/3}{(2/3)^2} = \frac{1}{3}\cdot\frac{9}{4} = \boxed{\mathbf{\frac{3}{4}}}.$$

7. Use term-by-term integration to find a power series for $f(x) = \ln(4 - x)$ centered at $a = 0$. What is its radius of convergence?

Solution: Differentiate, expand, integrate back.

$$f'(x) = \frac{-1}{4-x} = -\frac{1}{4} \cdot \frac{1}{1 - \frac{x}{4}} = -\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}, \qquad |x| < 4.$$

Integrate term by term:

$$\ln(4 - x) = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,4^{n+1}}.$$

Find C. At $x = 0$, $\ln 4 = C - 0$, so $C = \ln 4$:

$$\ln(4 - x) = \ln 4 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,4^{n+1}}.$$

Integrating doesn't change the radius, so $\mathbf{R = 4}$. (This is the distance from 0 to $x = 4$, where $\ln(4 - x)$ is undefined.)

8. Find the Taylor polynomial of degree 2 for $f(x) = \cos(3x)$ centered at $a = \pi$.

Solution:

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$	$f^{(n)}(\pi)/n!$
0	$\cos(3x)$	$\cos 3\pi = -1$	-1
1	$-3\sin(3x)$	$-3\sin 3\pi = 0$	0
2	$-9\cos(3x)$	$-9(-1) = 9$	$\frac{9}{2}$

$$P_2(x) = -1 + \frac{9}{2}(x - \pi)^2.$$

(Note the chain-rule factors of 3 and 9.)

9. (9.1) Find the Taylor series of $f(x) = e^x$ centered at $a = -1$.

Solution: Every derivative is e^x, so $f^{(n)}(-1) = e^{-1}$ for all n:

$$e^x = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{(x+1)^n}{e \cdot n!}, \qquad R = \infty.$$

(Equivalently: $e^x = e^{-1}e^{x+1}$, then substitute $u = x + 1$ into $e^u = \sum \frac{u^n}{n!}$.)

(9.2) Find the Taylor series of $f(x) = \dfrac{1}{x}$ centered at $a = -1$, and find its interval of convergence.

Solution: Rewrite the denominator in terms of $(x+1)$: $x = -1 + (x+1) = -\big(1 - (x+1)\big)$. So

$$\frac{1}{x} = \frac{-1}{1 - (x+1)} = -\sum_{n=0}^{\infty} (x+1)^n.$$

This converges when $|x + 1| < 1$. At both endpoints the terms don't go to 0.

Interval of convergence: $(-2,\ 0)$**.** This makes sense: the series can't reach past $x = 0$, where $\frac{1}{x}$ blows up.

Check: $f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$, so $\frac{f^{(n)}(-1)}{n!} = (-1)^n(-1)^{-n-1} = -1$ for every n.

10. Let $f(x) = (1+x^2)^{-1/3}$.

(10.1) Use the binomial series to write the Maclaurin series of f in Σ-notation.

Solution: The binomial series $(1+u)^r = \sum_{k=0}^{\infty} \binom{r}{k} u^k$, with $r = -\frac{1}{3}$ and $u = x^2$, gives

$$(1+x^2)^{-1/3} = \sum_{k=0}^{\infty} \binom{-1/3}{k} x^{2k}, \qquad |x| < 1.$$

Writing out the coefficient, each factor of $\binom{-1/3}{k}$ is negative, with numerators $1, 4, 7, \ldots, 3k-2$:

$$\binom{-1/3}{k} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\cdots\left(-\frac{3k-2}{3}\right)}{k!} = \frac{(-1)^k}{k!\,3^k} \cdot 1 \cdot 4 \cdot 7 \cdots (3k-2),$$

so

$$(1+x^2)^{-1/3} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\, 1 \cdot 4 \cdot 7 \cdots (3k-2)}{k!\,3^k}\, x^{2k}.$$

(10.2) Find the sixth Maclaurin polynomial $P_6(x)$ for f.

Solution: The powers are x^0, x^2, x^4, x^6, i.e. $k = 0, 1, 2, 3$:

$$\binom{-1/3}{1} = -\frac{1}{3}, \qquad \binom{-1/3}{2} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2} = \frac{2}{9}, \qquad \binom{-1/3}{3} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{6} = -\frac{14}{81}.$$

$$P_6(x) = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6.$$

11. Find a power series for $F(x) = \displaystyle\int_0^x e^{-t^2}\, dt$.

Solution: Substitute $u = -t^2$ into $e^u = \sum \frac{u^n}{n!}$:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}.$$

Integrate term by term from 0 to x, using $\int_0^x t^{2n}\, dt = \frac{x^{2n+1}}{2n+1}$:

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!} = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

There is no $+C$, because $F(0) = 0$. This converges for all x.

12. Find the first three nonzero terms of the Maclaurin series for $f(x) = \ln(\cos x)$.

Solution: Write $\cos x = 1 + u$, where

$$u = \cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

Then use $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots$, keeping powers up to x^6:

$$u^2 = \frac{x^4}{4} - \frac{x^6}{24} + \cdots, \qquad u^3 = -\frac{x^6}{8} + \cdots$$

Collect like powers:

- x^2: $-\frac{1}{2}$
- x^4: $\frac{1}{24} - \frac{1}{8} = -\frac{1}{12}$
- x^6: $-\frac{1}{720} + \frac{1}{48} - \frac{1}{24} = \frac{-1+15-30}{720} = -\frac{1}{45}$

$$\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots$$

Check: the derivative is $-\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right)$, which is the start of $-\tan x$, and $\frac{d}{dx}\ln(\cos x) = -\tan x$.