The Main Idea Behind Taylor's Theorem

A Proof Sketch via Iterated Integration by Parts

Table of contents

1 Overview 1

2 Base Step: Fundamental Theorem of Calculus 1

3 Step 1: First Integration by Parts 2

4 Step 2: Inductive Iteration 2

5 General Form: Taylor's Formula with Integral Remainder 3

6 Deriving the Lagrange Form of the Remainder 3

1 Overview

The core insight behind Taylor's theorem is that **iterated integration by parts accumulates successive derivative terms at the base point while packaging the cumulative approximation error into an exact integral remainder**.

---

2 Base Step: Fundamental Theorem of Calculus

Let $f \in C^{n+1}([a, x])$. By the Fundamental Theorem of Calculus:

$$f(x) = f(a) + \int_a^x f'(t)\, dt$$

---

3 Step 1: First Integration by Parts

To extract the linear derivative term, apply integration by parts to $\int_a^x f'(t)\,dt$.

A clever choice of antiderivative for $1 \cdot dt$ makes the boundary term vanish at the upper limit $t = x$:

- Let $u = f'(t) \implies du = f''(t)\,dt$
- Let $dv = dt \implies v = -(x - t)$

Evaluating the integral:

$$\int_a^x f'(t)\,dt = \Big[-f'(t)(x-t) \Big]_a^x - \int_a^x -(x-t)f''(t)\,dt$$

Because the upper evaluation at $t = x$ is zero:

$$\int_a^x f'(t)\,dt = f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt$$

Substituting back into the base equation yields:

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt$$

---

4 Step 2: Inductive Iteration

Repeat the integration by parts on the remainder integral $\int_a^x (x-t)f''(t)\,dt$:

- Let $u = f''(t) \implies du = f'''(t)\,dt$
- Let $dv = (x-t)\,dt \implies v = -\frac{(x-t)^2}{2}$

Evaluating:

$$\int_a^x (x-t)f''(t)\,dt = \left[-f''(t)\frac{(x-t)^2}{2}\right]_a^x + \int_a^x \frac{(x-t)^2}{2}f'''(t)\,dt$$

$$\int_a^x (x-t)f''(t)\,dt = \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2}f'''(t)\,dt$$

---

5 General Form: Taylor's Formula with Integral Remainder

Continuing this procedure n times yields:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where the exact remainder is given by the **Cauchy integral form**:

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

---

6 Deriving the Lagrange Form of the Remainder

Because $(x-t)^n$ does not change sign for t between a and x, we apply the **Mean Value Theorem for Integrals**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}\int_a^x (x-t)^n\,dt \quad \text{for some } \xi \in (a,x)$$

Evaluating the remaining polynomial integral:

$$\int_a^x (x-t)^n\,dt = \left[-\frac{(x-t)^{n+1}}{n+1}\right]_a^x = \frac{(x-a)^{n+1}}{n+1}$$

Substituting this back gives **Lagrange's form of the remainder**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

When $f \in C^\infty$ and $\lim_{n\to\infty} R_n(x) = 0$, the partial sums converge to the infinite Taylor series.